PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Determination of the Surface-Tension of Water by the Method of Jet Vibration

N. Bohr

Phil. Trans. R. Soc. Lond. A 1909 **209**, 281-317
doi: 10.1098/rsta.1909.0012

XII. *Determination of the Surface-Tension of Water by the Method of Jet Vibration.**

By N. Bohr, *Copenhagen.*

Communicated by Sir William Ramsay, *K.C.B., F.R.S.*

Received January 12,—Read January 21, 1909.

Introduction.

It has been shown that one of the most important and difficult questions in regard to the determination of the surface-tension of water is to produce a sufficiently pure surface, and in later investigations great importance has therefore been attached to this point.

In 1879 Lord Rayleigh,† however, indicated a method which solves the above-mentioned difficulty in a far more perfect manner than any other method used hitherto; this method makes it possible to determine the surface-tension of an almost perfectly fresh and constantly newly formed surface.

In the paper cited above, Lord Rayleigh has developed the theory of the vibrations of a jet of liquid under the influence of surface-tension, and as appears from this theory, it is possible to determine the surface-tension of a liquid when the velocity and cross-section of a jet of liquid, and the length of the waves formed on the jet, are known.

Lord Rayleigh has attached a series of experiments to the theoretical development. By these experiments, however, it was more especially intended to give illustrations of the theory rather than to give an exact determination of the surface-tension.

If, however, this is the problem, it is necessary to consider more closely some questions which are not discussed in Lord Rayleigh's investigation, for it is necessary

* Based on a response to Det Kongl. Danske Videnskabernes Selskabs (The Royal Danish Scientific Society's) Problem in Physics for 1905; delivered October 30, 1906; awarded the Society's Gold Medal. (The investigation has since been completed with a number of experiments.)

† Lord Rayleigh, 'Roy. Soc. Proc.,' vol. XXIX., p. 71, 1879.

to be sure, first, that the theoretical treatment is sufficiently developed, and secondly, that the phenomenon satisfies, to a sufficient degree, the assumptions on which the theoretical treatment rests.

The main purpose of the present investigation is to try to show how this can be done.

In spite of the great advantages of the above-mentioned method for the determination of surface-tension, it has, however, not been very much used. Except by Lord RAYLEIGH,* the method has till recently been used only by F. PICCARD† and G. MEYER‡ for relative measurements. During the completion of this investigation a treatise on this subject has been published by P. O. PEDERSEN.§

The Theory of the Vibrations of a Jet.

The theory of the vibrations of a jet of liquid about its cylindrical form of equilibrium has been developed by Lord RAYLEIGH for the case in which the amplitudes of the vibrations are infinitely small and the liquid has no viscosity.

The equations found by Lord RAYLEIGH can, when the amplitudes have small values and the viscosity coefficient is small, be considered as a good approximation; but if the equations are to be used for exact determination of the surface-tension, it is of importance to know how great the approximation is under the given circumstances. In the first part of this investigation we will therefore attempt to supplement the theory with corrections both for the influence of the finite amplitudes and for the viscosity.

Calculation of the Effect of the Viscosity.

Under the influence of the viscosity the jet will execute damped vibrations. If the problem is to find the law according to which the amplitudes decrease, this can, when the viscosity-coefficient is small, be done with approximation by a simple consideration of the energy dissipated. Some authors‖ are of opinion that the correction on the wave-length (time of vibration) due to the viscosity for a problem of this kind can be found directly from the logarithmic decrement of the wave-amplitudes δ by means of the formula $T_1 = T(1+\delta^2/4\pi^2)^{1/2}$, where T_1 is the time of vibration with damping, T is the time of vibration without it. This application of the formula given does not, however, seem to me to be correct. For the formula is established for a problem by which the only difference between the equation of motion

* RAYLEIGH, 'Roy. Soc. Proc.,' vol. XLVII., p. 281, 1890.

† PICCARD, 'Archives d. Sc. Phys. et Nat.' (3), XXIV., p. 561, 1890 (Genève).

‡ MEYER, 'WIED. Ann.,' LXVI., p. 523, 1898.

§ PEDERSEN, 'Phil. Trans. Roy. Soc.,' A, 207, p. 341, 1907.

‖ P. O. PEDERSEN (*loc. cit.*, p. 346); and PH. LENARD ('WIED. Ann.,' XXX., p. 239, 1887) in his paper about the analogous problem, the vibrations of a drop.

in normal co-ordinates for the conservative system $\left(a\dfrac{\partial q^2}{\partial t^2}+cq=0.\right.$ One degree of freedom. Small oscillations. Free motion$\left.\right)$ and the equation of motion for the dissipative system consists in the addition of a frictional term $\left(a\dfrac{\partial q^2}{\partial t^2}+b\dfrac{\partial q}{\partial t}+cq=0\right)$.

In the present problem—as in all fluid-problems in which a velocity-potential exists for the conservative system, but not for the dissipative one—the coefficient of inertia a will not be the same for the two systems, since a in the dissipative system is dependent on the coefficient of viscosity.

As will be seen from what follows, the correction on the wave-length will not be proportional to δ^2 but to $\delta^{3/2}$.

In order to find the variation of the wave-length due to the viscosity, the problem must be treated in greater detail. Such an investigation is given by Lord Rayleigh* for the vibrations of a cylinder of viscous fluid under capillary force, in the case where the original symmetry about the axis of the cylinder is maintained. In the development to be found in the paper cited above, the assumption mentioned (the symmetry) is, however, from the outset used in such a manner that the calculation cannot be extended to treat the more general vibrations which will be mentioned here. The result of our development does not include the problem investigated by Lord Rayleigh, as, in order to simplify the calculation, special precautions relative to the limiting case ($n=0$) are not taken.

The general equations of motion of an incompressible viscous fluid, unaffected by extraneous forces, are

$$\mu\nabla u-\rho\frac{\mathrm{D}u}{\mathrm{D}t}=\frac{\partial p}{\partial x},\quad \mu\nabla v-\rho\frac{\mathrm{D}v}{\mathrm{D}t}=\frac{\partial p}{\partial y},\quad \mu\nabla w-\rho\frac{\mathrm{D}w}{\mathrm{D}t}=\frac{\partial p}{\partial z}\quad . \quad . \quad . \quad (1)$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which u, v, w are the components of the velocity, p the pressure, ρ the density, μ the coefficient of viscosity, and

$$\nabla=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},\quad \frac{\mathrm{D}}{\mathrm{D}t}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}.$$

In the problem in question the motion will be steady. Putting $w=c+\omega$, and supposing that u, v, and ω have the form $f(x,y)\,e^{ibz}$, and that u, v, and ω are so small that products of them and quantities of the same order of magnitude can be neglected in the calculations, we get from the equations (1)

$$\left(\nabla-ib\frac{c\rho}{\mu}\right)u=\frac{1}{\mu}\frac{\partial p}{\partial x},\quad \left(\nabla-ib\frac{c\rho}{\mu}\right)v=\frac{1}{\mu}\frac{\partial p}{\partial y},\quad \left(\nabla-ib\frac{c\rho}{\mu}\right)\omega=\frac{1}{\mu}\frac{\partial p}{\partial z};\quad . \quad (3)$$

* Lord Rayleigh, 'Phil. Mag.,' XXXIV., p. 145, 1892.

from (3) and (2) it follows that

$$\nabla p = 0. \quad \ldots \quad (4)$$

Putting

$$u = \frac{i}{cb\rho}\frac{\partial p}{\partial x} + u_1, \quad v = \frac{i}{cb\rho}\frac{\partial p}{\partial y} + v_1, \quad \omega = \frac{i}{cb\rho}\frac{\partial p}{\partial z} + \omega_1 \quad \ldots \quad (5)$$

we get

$$\left(\nabla - ib\frac{c\rho}{\mu}\right)u_1 = 0, \quad \left(\nabla - ib\frac{c\rho}{\mu}\right)v_1 = 0, \quad \left(\nabla - ib\frac{c\rho}{\mu}\right)\omega_1 = 0, \quad \ldots \quad (6)$$

and

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial \omega_1}{\partial z} = 0. \quad \ldots \quad (7)$$

Now introducing polar co-ordinates r and ϑ ($x = r\cos\vartheta$, $y = r\sin\vartheta$), and the radial and tangential velocity α and β, we get, by help of the following relations,

$$\left.\begin{aligned} u &= \alpha\cos\vartheta - \beta\sin\vartheta, \quad u_1 = \alpha_1\cos\vartheta - \beta_1\sin\vartheta, \quad \frac{\partial}{\partial x} = \cos\vartheta\frac{\partial}{\partial r} - \sin\vartheta\frac{1}{r}\frac{\partial}{\partial \vartheta}, \\ v &= \alpha\sin\vartheta + \beta\cos\vartheta, \quad v_1 = \alpha_1\sin\vartheta + \beta_1\cos\vartheta, \quad \frac{\partial}{\partial y} = \sin\vartheta\frac{\partial}{\partial r} + \cos\vartheta\frac{1}{r}\frac{\partial}{\partial \vartheta}, \end{aligned}\right\} \quad . \quad (8)$$

from (5)

$$\alpha = \frac{i}{cb\rho}\frac{\partial p}{\partial r} + \alpha_1, \quad \beta = \frac{i}{cb\rho}\frac{1}{r}\frac{\partial p}{\partial \vartheta} + \beta_1, \quad \ldots \quad (9)$$

and from (6) and (7), considering $\nabla = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2} + \frac{\partial^2}{\partial z^2}$,

$$\left(\nabla - ib\frac{c\rho}{\mu}\right)\alpha_1 - \frac{\alpha_1}{r^2} - \frac{2}{r^2}\frac{\partial \beta_1}{\partial \vartheta} = 0, \quad \left(\nabla - ib\frac{c\rho}{\mu}\right)\beta_1 - \frac{\beta_1}{r^2} + \frac{2}{r^2}\frac{\partial \alpha_1}{\partial \vartheta} = 0, \quad \ldots \quad (10)$$

and

$$\frac{\partial \alpha_1}{\partial r} + \frac{\alpha_1}{r} + \frac{1}{r}\frac{\partial \beta_1}{\partial \vartheta} + \frac{\partial \omega_1}{\partial z} = 0. \quad \ldots \quad (11)$$

Now supposing that p, α, β, ω, and consequently α_1, β_1, ω_1, have the form $f(r)\,e^{in\vartheta + ibz}$, we get from (4)

$$\nabla p = \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} - p\left(\frac{n^2}{r^2} + b^2\right) = 0,$$

of which the solution, subject to the condition to be imposed when $r = 0$, is

$$p = \mathrm{A}\mathrm{J}_n(ibr)\,e^{in\vartheta + ibz}, \quad \ldots \quad (12)$$

in which J_n is the symbol of the BESSEL'S function of n^{th} order.

From (6) we get

$$\left(\nabla - ib\frac{c\rho}{\mu}\right)\omega_1 = \frac{\partial^2 \omega_1}{\partial r^2} + \frac{1}{r}\frac{\partial \omega_1}{\partial r} - \omega_1\left(\frac{n^2}{r^2} + d^2\right) = 0, \qquad \left(d^2 = b^2 + ib\frac{c\rho}{\mu}\right), \quad . \quad (13)$$

which gives

$$\omega_1 = \mathrm{B}\mathrm{J}_n(idr)\,e^{in\vartheta + ibz}. \quad \ldots \quad (14)$$

Eliminating β_1 from (10) and (11), we get

$$r\left(\nabla - ib\frac{c\rho}{\mu}\right)\alpha_1 + 2\frac{\partial \alpha_1}{\partial r} + \frac{\alpha_1}{r} = -2\frac{\partial \omega_1}{\partial z},$$

which gives

$$\left(\nabla - ib\frac{c\rho}{\mu}\right)(r\alpha_1) = -2ib\mathrm{J}_n(idr)\,e^{in\vartheta+ibz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

We have, however,

$$\left(\nabla - ib\frac{c\rho}{\mu}\right)\left(r\frac{\partial}{\partial r}\right) = \left(r\frac{\partial}{\partial r}\right)\left(\nabla - ib\frac{c\rho}{\mu}\right) + 2\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{2}{r^2}\frac{\partial^2}{\partial \vartheta^2}$$

$$= \left(r\frac{\partial}{\partial r} + 2\right)\left(\nabla - ib\frac{c\rho}{\mu}\right) - 2\left(\frac{\partial^2}{\partial z^2} - ib\frac{c\rho}{\mu}\right),$$

whence we get

$$\left(\nabla - ib\frac{c\rho}{\mu}\right)\left[r\frac{\partial}{\partial r}\mathrm{J}_n(idr)e^{in\vartheta+ibz}\right] = 2\left(b^2 + ib\frac{c\rho}{\mu}\right)\mathrm{J}_n(idr)\,e^{in\vartheta+ibz} = 2d^2\mathrm{J}_n(idr)\,e^{in\vartheta+ibz}; \quad (16)$$

from (15) and (16) we get

$$\alpha_1 = \left[\frac{b}{d}\mathrm{BJ}'_n(idr) + \mathrm{C}\frac{1}{r}\mathrm{J}_n(idr)\right]e^{in\vartheta+ibz}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and from (11) we get

$$-\frac{1}{r}\frac{\partial \beta_1}{\partial \vartheta} = \frac{\partial \alpha_1}{\partial r} + \frac{\alpha_1}{r} + \frac{\partial \omega_1}{\partial z}$$

$$= \left\{\mathrm{B}\left[ib\mathrm{J}''_n(idr) + \frac{b}{d}\frac{1}{r}\mathrm{J}'_n(idr) + ib\mathrm{J}_n(idr)\right] + \mathrm{C}id\frac{1}{r}\mathrm{J}'_n(idr)\right\}e^{in\vartheta+ibz}. \quad . \quad (18)$$

With the help of the relation

$$\mathrm{J}''_n(x) + \frac{1}{x}\mathrm{J}'_n(x) + \left(1 - \frac{n^2}{x^2}\right)\mathrm{J}_n(x) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

(18) gives

$$\beta_1 = \left[\mathrm{B}\frac{nb}{d^2}\frac{1}{r}\mathrm{J}_n(idr) - \mathrm{C}\frac{d}{n}\mathrm{J}'_n(idr)\right]e^{in\vartheta+ibz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Introducing in the equations (9) and (5) the values of p, α_1, β_1, and ω_1, found by (12), (14), (17), and (20), we get

$$\alpha = \left[-\mathrm{A}\frac{1}{c\rho}\mathrm{J}'_n(ibr) + \mathrm{B}\frac{b}{d}\mathrm{J}'_n(idr) + \mathrm{C}\frac{1}{r}\mathrm{J}_n(idr)\right]e^{in\vartheta+ibz},$$

$$\beta = \left[-\mathrm{A}\frac{n}{bc\rho}\frac{1}{r}\mathrm{J}_n(ibr) + \mathrm{B}\frac{bn}{d^2}\frac{1}{r}\mathrm{J}_n(idr) - \mathrm{C}\frac{d}{n}\mathrm{J}'_n(idr)\right]e^{in\vartheta+ibz}, \quad . \quad . \quad . \quad (21)$$

$$w = c + \omega = c + \left[-\mathrm{A}\frac{1}{e\rho}\mathrm{J}_n(ibr) + \mathrm{BJ}_n(idr)\right]e^{in\vartheta+ibz}.$$

Let us suppose that the equation of the surface is

$$r-a = \zeta = \mathrm{D}e^{in\vartheta+ibz}.$$

The general surface-condition gives

$$\frac{\mathrm{D}}{\mathrm{D}t}(r-a-\zeta) = \left(\alpha\frac{\partial}{\partial r}+\frac{\beta}{r}\frac{\partial}{\partial\vartheta}+w\frac{\partial}{\partial z}\right)(r-a-\zeta) = 0,$$

whence we get, neglecting quantities of the same order of magnitude as above,

$$\alpha-c\frac{\partial\zeta}{\partial z} = 0, \quad \zeta = -\frac{i}{cb}\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

In the same manner we get further, if the principal radii of curvature are R_1 and R_2,

$$\frac{1}{\mathrm{R}_1}+\frac{1}{\mathrm{R}_2} = \frac{1}{a}-\frac{\zeta}{a^2}-\frac{1}{a^2}\frac{\partial^2\zeta}{\partial\vartheta^2}-\frac{\partial^2\zeta}{\partial z^2} = \frac{1}{a}-\alpha\frac{i\,(n^2-1+b^2a^2)}{a^2cb}. \quad . \quad . \quad . \quad (23)$$

Let $\mathrm{P}r$, $\mathrm{P}\vartheta$, $\mathrm{P}z$ be respectively the radial, tangential, and axial component of the traction, per unit area, exerted by the viscous fluid across a surface-element perpendicular to radius-vector. Taking the radius-vector concerned as X-axis, and using the notation generally employed, we have

$$\mathrm{P}r = p_{x,x} = -p+2\mu\frac{\partial u}{\partial x}, \quad \mathrm{P}\vartheta = p_{x,y} = \mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right), \quad \mathrm{P}z = p_{x,z} = \mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right).$$

Using the relations (8), and after the differentiation setting $\vartheta = 0$, we get

$$\mathrm{P}r = -p+2\mu\frac{\partial\alpha}{\partial r}, \quad \mathrm{P}\vartheta = \mu\left(\frac{\partial\beta}{\partial r}+\frac{1}{r}\frac{\partial\alpha}{\partial\vartheta}-\frac{\beta}{r}\right), \quad \mathrm{P}z = \mu\left(\frac{\partial\alpha}{\partial z}+\frac{\partial w}{\partial r}\right). \quad . \quad (24)$$

Calling the surface-tension T and assuming that there is no "superficial viscosity," the dynamical surface-conditions will be, using the same rate of approximation as before,

$$\mathrm{T}\left(\frac{1}{\mathrm{R}_1}+\frac{1}{\mathrm{R}_2}\right)+\mathrm{P}r = \text{const.}, \quad \mathrm{P}\vartheta = 0, \quad \mathrm{P}z = 0\,; \quad . \quad . \quad . \quad . \quad . \quad (25)$$

from (25) we get, using (23) and (24),

$$\left[-\mathrm{T}\alpha\frac{i\,(n^2-1+a^2b^2)}{a^2cb}-p+2\mu\frac{\partial\alpha}{\partial r}\right]_{r=a} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$\left(\frac{1}{r}\frac{\partial\alpha}{\partial\vartheta}+\frac{\partial\beta}{\partial r}-\frac{\beta}{r}\right)_{r=a} = 0, \quad \left(\frac{\partial\alpha}{\partial z}+\frac{\partial w}{\partial r}\right)_{r=a} = 0. \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Introducing in these conditions the values of p, α, β, w found by (12) and (21), we get, after the elimination of B/A and C/A, an equation for the determination of b.

As these calculations will be rather long and the result unmanageable, we will not perform the elimination exactly, but only with an approximation which takes regard of the application of the results.

In the experiments the numerical value of iab will be small—the wave-length large in comparison to the diameter of the jet—and the numerical value of iad great—the coefficient of viscosity small—(in all the experiments $|iab| < 0{\cdot}24$ and $|iad| > 20$).

For every value of x we have

$$J_n(x) = \frac{x^n}{2^n [n]} - \frac{x^{n+2}}{2^{n+2}[n+1]} + \frac{x^{n+4}}{1.2.2^{n+4}[n+2]} - \ldots \quad . \quad . \quad . \quad (28)$$

The series converges rapidly for small numerical values of x, but very slowly for great values. From (28) it follows that

$$J'_n(x) = \frac{n}{x} J_n(x) \left[1 - \frac{x^2}{2n(n+1)} - \frac{x^4}{2^3 n (n+1)^2 (n+2)} \cdots \right],$$

and further, by (19), that

$$J''_n(x) = \frac{n(n-1)}{x^2} J_n(x) \left[1 - \frac{x^2 (2n+1)}{2(n-1)n(n+1)} + \frac{x^4}{2^3 (n-1) n (n+1)^2 (n+2)} \cdots \right].$$

Referring to the above, by the calculation of the frictional terms in the equation for the determination of b, we will therefore put

$$J'_n(iab) = -\frac{in}{ab} J_n(iab) \left[1 + \frac{a^2 b^2}{2n(n+1)}\right]$$

$$\text{and} \quad J''_n(iab) = -\frac{n(n-1)}{a^2 b^2} J_n(iab) \left[1 + \frac{a^2 b^2 (2n+1)}{2(n-1)n(n+1)}\right]. \quad . \quad . \quad (29)$$

For calculating $J_n(x)$ for great values of x the asymptotical expression

$$J_n(x) \backsim (2\pi x)^{-\frac{1}{2}} \left\{ [P_n(x) + iQ_n(x)] e^{i\left(x - \frac{2n+1}{4}\pi\right)} + [P_n(x) - iQ_n(x)] e^{-i\left(x - \frac{2n+1}{4}\pi\right)} \right\} \quad . \quad (30)$$

is used, in which

$$P_n(x) = 1 - \frac{(4n^2-1^2)(4n^2-3^2)}{1.2\,(8x)^2} + \frac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)(4n^2-7^2)}{1.2.3.4\,(8x)^4} - \ldots,$$

and

$$Q_n(x) = \frac{(4n^2-1^2)}{8x} - \frac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)}{1.2.3\,(8x)^3} + \ldots.$$

A few terms of the formula (30), which is only correct when the real component of x is positive, will for a great numerical value of x give an excellent approximation for $J_n(x)$. By our application of the formula (30), x can be given the form $\alpha - i b$, where both a and b are great positive quantities. Thereby the term with e^{ix} will be quite predominant; we therefore get, neglecting the term with e^{-ix},

$$J'_n(x) = iJ_n(x)\left[1+\frac{i}{2x}-\frac{4n^2-1}{8x^2}+\frac{i(4n^2-1)}{8x^3}\dots\right],$$

and further, by (19),

$$J''_n(x) = -J_n(x)\left[1+\frac{i}{x}-\frac{2n^2+1}{2x^2}-\frac{i(4n^2-1)}{8x^3}\dots\right].$$

In the following calculations we will therefore put

$$J'_n(iad) = iJ_n(iad)\left(1+\frac{1}{2ad}+\frac{4n^2-1}{8a^2d^2}\right) \text{ and } J''_n(iad) = \div J_n(iad)\left(1+\frac{1}{ad}+\frac{2n^2+1}{2a^2d^2}\right). \quad (31)$$

From (27) we get now, using (29) and (31),

$$A\frac{1}{c\rho}J_n(iab)\frac{2n(n-1)}{a^2b}\left[1+\frac{a^2b^2}{2(n-1)(n+1)}\right]$$
$$+BJ_n(iad)\frac{2nb}{ad}\left(1+\frac{3}{2ad}+\frac{4n^2-1}{8a^2d^2}\right)-CJ_n(iad)\frac{id^2}{n}\left(1+\frac{2}{ad}+\frac{2n^2+1}{a^2d^2}\right)=0, \quad . \quad (32)$$

and

$$A\frac{1}{c\rho}J_n(iab)\frac{2n}{a}\left[1+\frac{a^2b^2}{2n(n+1)}\right]+BJ_n(iad)\,d\left(1+\frac{1}{2ad}+\frac{4n^2-1}{8a^2d^2}\right)-CJ_n(iad)\frac{ib}{a}=0; \quad (33)$$

from (32) and (33) we get

$$BJ_n(iad) = \div A\frac{1}{c\rho}J_n(iab)\frac{2n}{ad}\left[1+\frac{a^2b^2}{2n(n+1)}\right]\left(1-\frac{1}{2ad}-\frac{12n^2-8n-3}{8a^2d^2}\right)$$

and $\quad . \quad (34)$

$$CJ_n(iad) = \div A\frac{1}{c\rho}J_n(iab)\frac{i2n^2(n-1)}{a^2d^2b}\left[1+\frac{a^2b^2}{2(n^2-1)}\right]\left(1-\frac{2}{ad}-\frac{2n^2-3}{a^2d^2}\right)$$

From (26) we get now, using (12), (21), (29), (31), (34) and (13),

$$b^2-ib\frac{\mu}{\rho}\frac{4n(n-1)}{a^2c}\left[1+\frac{a^2b^2}{n(n-1)}\right]\left[1+\frac{n-1}{ad}+\frac{(n-1)(2n-3)}{2a^2d^2}\right]$$
$$-T\frac{ibaJ'_n(iab)}{\rho c^2a^3J_n(iab)}(n^2-1+a^2b^2)=0. \quad . \quad . \quad (35)$$

Putting $\mu = 0$ in (35), we get the solution of Lord RAYLEIGH,*

$$b_0^2 = T\frac{iab_0J'_n(iab_0)}{\rho c^2a^3J_n(iab_0)}(n^2-1+a^2b_0^2)$$
$$=\frac{T(n^3-n)}{\rho c^2a^3}\left[1+\frac{(3n-1)a^2b_0^2}{2n(n^2-1)}+\frac{3(n+3)a^4b_0^4}{8n(n-1)(n+1)^2(n+2)}+\dots\right]. \quad . \quad (36)$$

In the following we will denote the positive root of this equation by k_0.

* Lord RAYLEIGH, 'Roy. Soc. Proc.,' vol. XXIX., p. 94, 1879.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

From (35) and (36) we get, with the same approximation as used hitherto,

$$b^2 - ib\frac{\mu}{\rho}\frac{4n(n-1)}{a^2c}\left[1+\frac{(5n+1)a^2k_0^2}{2n(n^2-1)}\right]\left[1+\frac{n-1}{ad}+\frac{(n-1)(2n-3)}{2a^2d^2}\right]-k_0^2 = 0. \quad . \quad (37)$$

With the same approximation it will be permissible, by use of (13), to put in (37)

$$iad = ia\left(ik_0\frac{c\rho}{\mu}\right)^{1/2} = (1-i)\left(\frac{a^2k_0c\rho}{2\mu}\right)^{1/2},$$

choosing the sign for iad so that the real component is positive [see (30)].

The equation (37) now becomes

$$b^2 - ib\frac{\mu}{\rho}\frac{4n(n-1)}{a^2c}\left[1+\frac{(5n+1)a^2k_0^2}{2n(n^2-1)}\right]$$
$$\left[1-(1-i)\frac{n-1}{2}\left(\frac{2\mu}{\rho ca^2k_0}\right)^{1/2}-i\frac{(n-1)(2n-3)}{4}\left(\frac{2\mu}{\rho ca^2k_0}\right)\right]-k_0^2 = 0. \quad . \quad (38)$$

Now solving (38) with regard to b, and setting $b = k+i\epsilon$, we get

$$k = k_0\left[1-\frac{n(n-1)^2}{2}\left(\frac{2\mu}{\rho ca^2k_0}\right)^{3/2}-\frac{3n(n-1)^2}{4}\left(\frac{2\mu}{\rho ca^2k_0}\right)^2\right] \quad . \quad . \quad . \quad . \quad (39)$$

and

$$\epsilon = \frac{\mu}{\rho}\frac{2n(n-1)}{a^2c}\left[1+\frac{(5n+1)a^2k_0^2}{2n(n^2-1)}\right]\left[1-\frac{n-1}{2}\left(\frac{2\mu}{\rho ca^2k_0}\right)^{1/2}\right]. \quad . \quad . \quad . \quad (40)$$

As all the equations used are linear, it will be seen that the physical meaning of the above calculation is the proof of the existence of a real fluid-motion corresponding to a surface of the form

$$r = a+be^{-\epsilon z}\cos kz \cos n\vartheta,$$

where k and ϵ are expressed by the equations (39) and (40).

The correction, which on account of the effect of the viscosity is to be introduced in the expression for the surface-tension, can be found by (36) and (39); we get

$$T = k^2\frac{\rho c^2a^3J_n(iak)}{iakJ'_n(iak)(n^2-1+a^2k^2)}\left[1+n(n-1)^2\left(\frac{2\mu}{\rho ca^2k}\right)^{3/2}+\frac{3n(n-1)^2}{2}\left(\frac{2\mu}{\rho ca^2k}\right)^2\right]. \quad . \quad (41)$$

Calculation of the Influence of the Finite Wave-amplitudes.

We will now calculate the correction of the wave-length due to the magnitude of the wave-amplitudes. The method of approximation which will be used is on the principle indicated by G. G. Stokes.*

The following calculation will only treat of the vibrations in two dimensions of a fluid-cylinder without viscosity. The problem in three dimensions could be treated

* G. G. Stokes, 'Camb. Trans.,' VIII., p. 441, 1847.

in a corresponding manner; but the calculations would in this case be very extensive, and, with regard to the present investigation, it would not be of any practical importance. Using jets the diameter of which is small in proportion to the wave-length, the motion will differ so little from the motion in two dimensions that the small correction of the wave-length due to the finite values of the amplitudes can be considered the same in both cases.

In the present problem the existence of a velocity-potential ϕ can be supposed. Using polar co-ordinates and calling the radial and tangential velocity respectively α and β, we get

$$\alpha = -\frac{\partial\phi}{\partial r}, \quad \beta = -\frac{1}{r}\frac{\partial\phi}{\partial\vartheta}.$$

Considering the fluid as incompressible, we get

$$0 = \frac{\partial\alpha}{\partial r} + \frac{\alpha}{r} + \frac{1}{r}\frac{\partial\beta}{\partial\vartheta} = -\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The solution of (1), subject to the condition that the velocity shall be finite for $r = 0$, can be written (n being a positive integer)

$$\phi = \Sigma\Sigma A_{n,q}\, r^n \cos(n\vartheta + \tau_n)\sin(qt + \epsilon_q). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The equation of the surface can be written

$$r = a + \zeta, \quad \zeta = \psi(\vartheta, t).$$

The surface-conditions are, using the same notations as above and calling the radius of curvature of the surface R,

$$\frac{D}{Dt}(a+\zeta-r) = \left(\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial r} + \frac{\beta}{r}\frac{\partial}{\partial\vartheta}\right)(a+\zeta-r) \quad \text{and} \quad p - \frac{T}{R} = 0. \quad . \quad . \quad . \quad (3)$$

From (3) we get

$$\left(\frac{\partial\zeta}{\partial t} - \frac{1}{r^2}\frac{\partial\phi}{\partial\vartheta}\frac{\partial\zeta}{\partial\vartheta} + \frac{\partial\phi}{\partial r}\right)_{r=a+\zeta} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$\rho\left\{\frac{\partial\phi}{\partial t} - \tfrac{1}{2}\left[\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\vartheta}\right)^2\right]\right\}_{r=a+\zeta}$$
$$-T\left[(a+\zeta)^2 + 2\left(\frac{\partial\zeta}{\partial\vartheta}\right)^2 - (a+\zeta)\frac{\partial^2\zeta}{\partial\vartheta^2}\right]\left[(a+\zeta)^2 + \left(\frac{\partial\zeta}{\partial\vartheta}\right)^2\right]^{-3/2} + F(t) = 0. \quad . \quad (5)$$

Considering only small vibrations of the surface about the position of equilibrium $r = a$, ζ is a small quantity which we will consider as being of the first order. From (2), (4), and (5) it can be seen that ϕ must also be of the first order, when $F(t)$ is defined in such a manner that ϕ does not contain terms independent of r or ϑ.

From the equations (4) and (5) we get, by help of TAYLOR's theorem,

$$\frac{\partial \zeta}{\partial t}+\left[\left(1+\zeta\frac{\partial}{\partial r}+\frac{\zeta^2}{2}\frac{\partial^2}{\partial r^2}\cdots\right)\left(\frac{\partial\phi}{\partial r}-\frac{1}{r^2}\frac{\partial\phi}{\partial\vartheta}\frac{\partial\zeta}{\partial\vartheta}\right)\right]_{r=a}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$\rho\left\{\left(1+\zeta\frac{\partial}{\partial r}+\frac{\zeta^2}{2}\frac{\partial^2}{\partial r^2}\cdots\right)\left[\frac{\partial\phi}{\partial t}-\frac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^2-\frac{1}{2r^2}\left(\frac{\partial\phi}{\partial\vartheta}\right)^2\right]\right\}_{r=a}$$

$$-\mathrm{T}\left[(a+\zeta)^2+2\left(\frac{\partial\zeta}{\partial\vartheta}\right)^2-(a+\zeta)\frac{\partial^2\zeta}{\partial\vartheta^2}\right]\left[(a+\zeta)^2+\left(\frac{\partial\zeta}{\partial\vartheta}\right)^2\right]^{-3/2}+\mathrm{F}(t)=0. \quad . \quad . \quad (7)$$

From (2), (6), and (7) ζ can be found except for a constant, which can be determined by the condition

$$\int_0^{2\pi}\int_0^{a+\zeta} r\,dr\,d\vartheta=\int_0^{2\pi}\tfrac{1}{2}(a+\zeta)^2\,d\vartheta=\pi a^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

First Approximation.

(Solution of the problem neglecting all terms of higher order than the first.)

From (6) and (7) we get

$$\frac{\partial\zeta}{\partial t}+\left[\frac{\partial\phi}{\partial r}\right]_{r=a}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and

$$\rho\left[\frac{\partial\phi}{\partial t}\right]_{r=a}-\mathrm{T}\left(\frac{1}{a}-\frac{\zeta}{a^2}-\frac{1}{a^2}\frac{\partial^2\zeta}{\partial\vartheta^2}\right)+\mathrm{F}(t)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Eliminating ζ from (9) and (10), we get

$$\left[\rho\frac{\partial^2\phi}{\partial t^2}-\frac{\mathrm{T}}{a^2}\left(\frac{\partial\phi}{\partial r}+\frac{\partial^3\phi}{\partial r\,\partial\vartheta^2}\right)\right]_{r=a}+\mathrm{F}'(t)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Putting $\mathrm{F}'(t)=0$, (11) will be satisfied by

$$\phi=\mathrm{A}r^n\cos n\vartheta\sin qt \qquad \text{when} \qquad q^2=\frac{\mathrm{T}}{\rho a^3}(n^3-n). \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Introducing this in (9), we get

$$\frac{\partial\zeta}{\partial t}=-na^{n-1}\mathrm{A}\cos n\vartheta\sin qt, \qquad \zeta=\frac{n}{q}a^{n-1}\mathrm{A}\cos n\vartheta\cos qt+f(\vartheta). \quad . \quad . \quad (13)$$

From (10) we get, using (12) and (13),

$$f(\vartheta)+f''(\vartheta)=\text{const.},$$

which is satisfied by

$$f(\vartheta)=\mathrm{C}.$$

From (8) we get in this case

$$\mathrm{C}=0.$$

To a first approximation we get as the general form of the vibrations

$$r = a + \Sigma b_n \cos(n\vartheta + \tau_n)\cos(q_n t + \epsilon_n), \qquad \text{where} \qquad q_n^2 = \frac{T}{\rho a^3}(n^3 - n).$$

As to the higher approximations, it is not possible to find the form for the general vibrations in a corresponding manner, the single types of vibration only being independent of each other to a first approximation.

We shall now determine the next approximations of the pure periodical type of vibration, of which the first approximation is defined by

$$\phi = \mathrm{A}r^n \cos n\vartheta \sin qt, \qquad \zeta = \frac{n}{q}a^{n-1}\mathrm{A}\cos n\vartheta \cos qt, \qquad q^2 = \frac{T}{\rho a^3}(n^3 - n). \quad (14)$$

Second Approximation.

From (6) and (7) we get

$$\frac{\partial\zeta}{\partial t} + \left[\frac{\partial\phi}{\partial r} + \zeta\frac{\partial^2\phi}{\partial r^2} - \frac{1}{r^2}\frac{\partial\phi}{\partial\vartheta}\frac{\partial\zeta}{\partial\vartheta}\right]_{r=a} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and

$$\rho\left[\frac{\partial\phi}{\partial t} + \zeta\frac{\partial^2\phi}{\partial r\,\partial t} - \frac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^2 - \frac{1}{2r^2}\left(\frac{\partial\phi}{\partial\vartheta}\right)^2\right]_{r=a}$$

$$-\mathrm{T}\left[\frac{1}{a} - \frac{\zeta}{a^2} - \frac{1}{a^2}\frac{\partial^2\zeta}{\partial\vartheta^2} + \frac{\zeta^2}{a^3} + \frac{1}{2a^3}\left(\frac{\partial\zeta}{\partial\vartheta}\right)^2 + \frac{2\zeta}{a^3}\frac{\partial^2\zeta}{\partial\vartheta^2}\right] + \mathrm{F}(t) = 0. \quad . \quad . \quad (16)$$

Introducing the values of ϕ, ζ, q, defined by (14), in the terms of second order, we get

$$\frac{\partial\zeta}{\partial t} + \left[\frac{\partial\phi}{\partial r}\right]_{r=a} = -\frac{n^2(2n-1)}{4q}a^{2n-3}\mathrm{A}^2\cos 2n\vartheta \sin 2qt + \frac{n^2}{4q}a^{2n-3}\mathrm{A}^2\sin 2qt, \quad . \quad (17)$$

and

$$\rho\left[\frac{\partial\phi}{\partial t}\right]_{r=a} - \mathrm{T}\left(\frac{1}{a} - \frac{\zeta}{a^2} - \frac{1}{a^2}\frac{\partial^2\zeta}{\partial\vartheta^2}\right) + \mathrm{F}(t)$$

$$= -\rho\frac{n(2n+1)(n^2+2n-2)}{8(n^2-1)}a^{2n-2}\mathrm{A}^2(\cos 2n\vartheta \cos 2qt + \cos 2n\vartheta)$$

$$-\rho\frac{n(4n^3+3n^2-4n-2)}{8(n^2-1)}a^{2n-2}\mathrm{A}^2\cos 2qt - \rho\frac{n(3n^2-2)}{8(n^2-1)}a^{2n-2}\mathrm{A}^2. \quad . \quad . \quad (18)$$

Eliminating ζ from (17) and (18), we get

$$\left[\rho\frac{\partial^2\phi}{\partial t^2} - \frac{\mathrm{T}}{a^2}\left(\frac{\partial\phi}{\partial r} + \frac{\partial^3\phi}{\partial r\,\partial\vartheta^2}\right)\right]_{r=a} + \mathrm{F}'(t)$$

$$= -\rho\frac{3qn(n-1)(2n+1)}{4(n+1)}a^{2n-2}\mathrm{A}^2\cos 2n\vartheta \sin 2qt + \frac{\rho}{4}qn(4n+3)a^{2n-2}\mathrm{A}^2\sin 2qt. \quad (19)$$

Putting

$$F'(t) = \frac{\rho}{4} qn(4n+3) a^{2n-2} A^2 \sin 2qt,$$

(19) will be satisfied by

$$\phi = Ar^n \cos n\vartheta \sin qt - \frac{3n(n-1)^2(2n+1)}{8(2n^2+1)qa^2} A^2 r^{2n} \cos 2n\vartheta \sin 2qt, \quad . \quad . \quad (20)$$

when, as by first approximation,

$$q^2 = \frac{T}{\rho a^3}(n^3 - n). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Introducing this in (17), we get

$$\frac{\partial\zeta}{\partial t} = -na^{n-1} A \cos n\vartheta \sin qt + A^2 \frac{n^2}{4q} \frac{2n^3 - 7n^2 - 2n + 4}{2n^2+1} a^{2n-3} \cos 2n\vartheta \sin 2qt$$
$$+ A^2 \frac{n^2}{4q} a^{2n-3} \sin 2qt; \quad . \quad . \quad (22)$$

from (22) we get

$$\zeta = \frac{n}{q} A a^{n-1} \cos n\vartheta \cos qt - A^2 \frac{n^2}{8q^2} \frac{2n^3 - 7n^2 - 2n + 4}{2n^2+1} a^{2n-3} \cos 2n\vartheta \cos 2qt$$
$$- A^2 \frac{n^2}{8q^2} a^{2n-3} \cos 2qt + f(\vartheta). \quad . \quad (23)$$

Introducing in (18) the values found for ϕ, q, ζ, and $F'(t)$, we get

$$f(\vartheta) + f''(\vartheta) = -A^2 \frac{n^2}{8q^2}(2n+1)(n^2+2n-2) a^{2n-3} \cos 2n\vartheta + \text{const.},$$

which is satisfied by

$$f(\vartheta) = A^2 \frac{n^2}{8q^2} \frac{n^2+2n-2}{2n-1} a^{2n-3} \cos 2n\vartheta + C. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

By (8) we get in this case

$$C = -A^2 \frac{n^2}{8q^2} a^{2n-3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

From (23), (24), and (25) we get

$$\zeta = \frac{n}{q} A a^{n-1} \cos n\vartheta \cos qt - A^2 \frac{n^2}{8q^2} \frac{2n^3 - 7n^2 - 2n + 4}{2n^2+1} a^{2n-3} \cos 2n\vartheta \cos 2qt$$
$$+ A^2 \frac{n^2}{8q^2} \frac{n^2+2n-2}{2n-1} a^{2n-3} \cos 2n\vartheta - A^2 \frac{n^2}{8q^2} a^{2n-3} \cos 2qt - A^2 \frac{n^2}{8q^2} a^{2n-3}. \quad . \quad (26)$$

Third Approximation.

From (6) and (7) we get

$$\frac{\partial\zeta}{\partial t} + \left[\frac{\partial\phi}{\partial r} - \frac{1}{r^2}\frac{\partial\phi}{\partial\vartheta}\frac{\partial\zeta}{\partial\vartheta} + \zeta\frac{\partial^2\phi}{\partial r^2} + \frac{\zeta^2}{2}\frac{\partial^3\phi}{\partial r^3} + \frac{2\zeta}{r^3}\frac{\partial\phi}{\partial\vartheta}\frac{\partial\zeta}{\partial\vartheta} - \frac{\zeta}{r^2}\frac{\partial^2\phi}{\partial r\partial\vartheta}\frac{\partial\zeta}{\partial\vartheta}\right]_{r=a} = 0, \quad . \quad (27)$$

and

$$\rho\left[\frac{\partial\phi}{\partial t}+\zeta\frac{\partial^2\phi}{\partial r\partial t}+\frac{\zeta^2}{2}\frac{\partial^3\phi}{\partial r^2\partial t}-\frac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^2-\frac{1}{2r^2}\left(\frac{\partial\phi}{\partial\vartheta}\right)^2-\zeta\frac{\partial^2\phi}{\partial r^2}\frac{\partial\phi}{\partial r}-\frac{\zeta}{r^2}\frac{\partial^2\phi}{\partial r\partial\vartheta}\frac{\partial\phi}{\partial\vartheta}+\frac{\zeta}{r^3}\left(\frac{\partial\phi}{\partial\vartheta}\right)^2\right]_{r=a}$$
$$-\mathrm{T}\left[\frac{1}{a}-\frac{\zeta}{a^2}-\frac{1}{a^2}\frac{\partial^2\zeta}{\partial\vartheta^2}+\frac{\zeta^2}{a^3}+\frac{1}{2a^3}\left(\frac{\partial\zeta}{\partial\vartheta}\right)^2+\frac{2\zeta}{a^3}\frac{\partial^2\zeta}{\partial\vartheta^2}\right.$$
$$\left.-\frac{\zeta^3}{a^4}-\frac{3\zeta}{2a^4}\left(\frac{\partial\zeta}{\partial\vartheta}\right)^2-\frac{3\zeta^2}{a^4}\frac{\partial^2\zeta}{\partial\vartheta^2}+\frac{3}{2a^4}\frac{\partial^2\zeta}{\partial\vartheta^2}\left(\frac{\partial\zeta}{\partial\vartheta}\right)^2\right]+\mathrm{F}(t)=0. \quad . \quad . \quad . \quad . \quad (28)$$

Introducing the values of ϕ, ζ, q, defined by (20), (21), and (26), we get (in order not to make the calculations more extensive than necessary, we will only calculate the terms which have references to the determination of q)

$$\frac{\partial\zeta}{\partial t}+\left[\frac{\partial\phi}{\partial r}\right]_{r=a}=\frac{n^3(n^2-1)(28n^3-42n^2+35n-6)}{32q^2(2n^2+1)(2n-1)}\mathrm{A}^3a^{3n-5}\cos n\vartheta\sin qt$$
$$+\mathrm{P}_1\cos 2n\vartheta\sin 2qt+\mathrm{P}_2\sin 2qt+\mathrm{P}_3\cos 3n\vartheta\sin 3qt$$
$$+\mathrm{P}_4\cos 3n\vartheta\sin qt+\mathrm{P}_5\cos n\vartheta\sin 3qt \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and

$$\rho\left[\frac{\partial\phi}{\partial t}\right]_{r=a}-\mathrm{T}\left(\frac{1}{a}-\frac{\zeta}{a^2}-\frac{1}{a^2}\frac{\partial^2\zeta}{\partial\vartheta^2}\right)+\mathrm{F}(t)$$
$$=-\rho\frac{n^2(n^2-1)(40n^3-24n^2+65n-30)}{32q(2n^2+1)(2n-1)}\mathrm{A}^3a^{3n-4}\cos n\vartheta\cos qt$$
$$+\mathrm{Q}_1\cos 2n\vartheta\cos 2qt+\mathrm{Q}_2\cos 2n\vartheta+\mathrm{Q}_3\cos 2qt+\mathrm{Q}_4$$
$$+\mathrm{Q}_5\cos 3n\vartheta\cos 3qt+\mathrm{Q}_6\cos 3n\vartheta\cos qt+\mathrm{Q}_7\cos n\vartheta\cos 3qt. \quad . \quad (30)$$

Eliminating ζ from (29) and (30), we get

$$\left[\rho\frac{\partial^2\phi}{\partial t^2}-\frac{\mathrm{T}}{a^2}\left(\frac{\partial\phi}{\partial r}+\frac{\partial^3\phi}{\partial r\partial\vartheta^2}\right)\right]_{r=a}+\mathrm{F}'(t)$$
$$=\rho\frac{n^2(n^2-1)(34n^3-33n^2+50n-18)}{16(2n^2+1)(2n-1)}\mathrm{A}^3a^{3n-4}\cos n\vartheta\sin qt$$
$$+\mathrm{S}_1\cos 2n\vartheta\sin 2qt+\mathrm{S}_2\sin 2qt+\mathrm{S}_3\cos 3n\vartheta\sin 3qt$$
$$+\mathrm{S}_4\cos 3n\vartheta\sin qt+\mathrm{S}_5\cos n\vartheta\sin 3qt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Setting $\mathrm{F}'(t)=\mathrm{S}_2\sin 2qt$, (31) will be satisfied by

$$\mathrm{Q}=\mathrm{A}r^n\cos n\vartheta\sin qt+\mathrm{A}_1r^{2n}\cos 2n\vartheta\sin 2qt+\mathrm{A}_2r^{3n}\cos 3n\vartheta\sin 3qt$$
$$+\mathrm{A}_3r^{3n}\cos 3n\vartheta\sin qt+\mathrm{A}_4r^n\cos n\vartheta\sin 3qt, \quad . \quad (32)$$

when

$$q^2=\frac{\mathrm{T}}{a^3\rho}(n^3-n)\left[1-\mathrm{A}^2a^{2n-4}\frac{n^2(n^2-1)(34n^3-33n^2+50n-18)}{16q^2(2n^2+1)(2n-1)}\right]. \quad . \quad . \quad (33)$$

Proceeding in the same manner as in the second approximation, we get

$$\zeta = \mathrm{A}\frac{n}{q}a^{n-1}\left[1-\mathrm{A}^2\frac{n^2}{q^2}a^{2n-4}\frac{(n^2-1)(28n^3-42n^2+35n-6)}{32(2n^2+1)(2n-1)}\right]\cos n\vartheta\cos qt$$
$$+\mathrm{B}_1\cos 2n\vartheta\cos 2qt+\mathrm{B}_2\cos 2n\vartheta+\mathrm{B}_3\cos 2qt+\mathrm{B}_4$$
$$+\mathrm{B}_5\cos 3n\vartheta\cos 3qt+\mathrm{B}_6\cos 3n\vartheta\cos qt+\mathrm{B}_7\cos n\vartheta\cos 3qt, \quad . \quad (34)$$

where the coefficients B_1, B_2, B_3, B_4 are the same as in the second approximation, and the coefficients B_5, B_6, B_7 are of the same order as A^3.

As the result of the calculation we note [putting the coefficient of $\cos n\vartheta\cos qt$ in the equation (34) equal to b], that the surface of a fluid-cylinder, executing pure periodical vibrations in two dimensions, can be expressed by

$$r = a+b\cos n\vartheta\cos qt+\frac{b^2}{a}\left[-\frac{2n^3-7n^2-2n+4}{8(2n^2+1)}\cos 2n\vartheta\cos 2qt\right.$$
$$\left.+\frac{n^2+2n-2}{8(2n-1)}\cos 2n\vartheta-\frac{1}{8}\cos 2qt-\frac{1}{8}\right]+\frac{b^3}{a^2}(\ldots)+\ldots, \quad . \quad . \quad . \quad . \quad (35)$$

where

$$q^2 = \frac{\mathrm{T}}{\rho a^3}(n^3-n)\left[1-\frac{b^2}{a^2}\frac{(n^2-1)(34n^3-33n^2+50n-18)}{16(2n^2+1)(2n-1)}+\frac{b^4}{a^4}(\ldots)+\ldots\right].$$

In the experiments, the jets produced (stationary waves) will execute vibrations in three dimensions—the cross-section will not be the same at different points of the jet. If, however, the velocity of the jet c is so great that the wave-length λ is great in comparison with the diameter of the jet, the motion in the single cross-sections will differ very little from the motion in two dimensions, and the equation (35) can, therefore, also in this case give information about the form of the surface of the jet.

The complete solution in three dimensions can be expressed by

$$r = a+b\cos n\vartheta\cos kz+\mathrm{N}_1\frac{b^2}{a}\left[1+\alpha_{1,1}\left(\frac{a}{\lambda}\right)^2+\alpha_{1,2}\left(\frac{a}{\lambda}\right)^4+\ldots\right]\cos 2n\vartheta\cos 2kz$$
$$+\mathrm{N}_2\frac{b^2}{a}\left[1+\alpha_{2,1}\left(\frac{a}{\lambda}\right)^2+\ldots\right]\cos 2n\vartheta+\ldots$$

and

$$k^2=\frac{1}{c^2}\frac{\mathrm{T}}{\rho a^3}(n^3-n)\left[1+\beta_1\left(\frac{a}{\lambda}\right)^2+\beta_2\left(\frac{a}{\lambda}\right)^4+\ldots\right]\left\{1+\mathrm{M}_1\frac{b^2}{a^2}\left[1+\gamma_1\left(\frac{a}{\lambda}\right)^2+\ldots\right]+\mathrm{M}_2\frac{b^4}{a^4}(1+\ldots)+\ldots\right\},$$

where the constant N_1, N_2,... and M_1, M_2,... will be equal to the corresponding constants in the equations (35), putting in these equations $t=\frac{z}{c}$ and $q=2\pi\frac{\lambda}{c}=kc$.

Neglecting the corrections in the terms of higher order in b/a, we get, using the formula of Lord RAYLEIGH for the wave-length of infinitely small vibrations in three dimensions [see (36), p. 288], and for sake of simplicity putting $n = 2$, which corresponds to the experiments executed,

$$r = a + b \cos 2\vartheta \cos kz + \frac{b^2}{6a} \cos 4\vartheta \cos 2kz + \frac{b^2}{4a} \cos 4\vartheta - \frac{b^2}{8a} \cos 2kz - \frac{b^2}{8a} \dots \quad (36)$$

and

$$k^2 = \frac{\mathrm{T} iak\, \mathrm{J}'_2(iak)}{\rho c^2 a^3\, \mathrm{J}_2(iak)} (3 + a^2k^2)\left(1 - \frac{b^2}{a^2}\frac{37}{24}\right). \quad (37)$$

The equation (37) gives the correction on the wave-length sought.

The equation (36) permits some further applications.

Putting $z = 0$, we get

$$r = a - \frac{b^2}{4a} + b \cos 2\vartheta + \frac{5}{12}\frac{b^2}{a} \cos 4\vartheta \dots \quad (38)$$

(38) is the equation for the form of the orifice (supposing that the velocity, at every point of the cross-section of the jet at the orifice, has the same magnitude and direction), when the jet is to execute pure periodical vibrations. We see from this that the opinion of P. O. PEDERSEN,* according to which a jet issuing from an orifice of the form $(r = \alpha + \beta \cos 2\vartheta)$ must be expected to execute much purer vibrations than a jet from an elliptical orifice $\left(r = \alpha + \beta \cos 2\vartheta + \frac{3}{4}\frac{\beta^2}{\alpha} \cos 4\vartheta \dots\right)$, is not correct.

Putting $\vartheta = 0$, we get

$$r = a + \frac{b^2}{8a} + b \cos kz + \frac{1}{24}\frac{b^2}{a} \cos 2kz \dots \quad (39)$$

(39) is the equation for the wave-profile, formed by intersecting the surface of the jet by one of the two perpendicular planes of symmetry. Maximum- and minimum-values of r are obtained by putting in (39) respectively $z = 2n\frac{\pi}{k}$ and $z = (2n+1)\frac{\pi}{k}$. We thus get

$$\tfrac{1}{2}(r_{\text{max.}} + r_{\text{min.}}) = a\left(1 + \frac{1}{6}\frac{b^2}{a^2}\right) \quad \text{and} \quad \tfrac{1}{2}(r_{\text{max.}} - r_{\text{min.}}) = b \,. \quad (40)$$

These formulas will be used in the measuring of the jets.

Calculation of the Effect of the surrounding Air.

We have hitherto neglected the density of the air.† A sufficient approximation of the small correction on the wave-length, due to the inertia of the air, is however very simply obtained by the following calculation regarding infinitely small vibrations in two dimensions of a cylindrical surface, separating two fluids of different density.

* P. O. PEDERSEN, *loc. cit.*, p. 365.

† Lord RAYLEIGH ('Phil. Mag.,' XXXIV., p. 177, 1892) has investigated the corresponding problem in the case where the symmetry about the axis of the fluid-cylinder is maintained during the vibrations.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Considering the fluids as inviscid, we can suppose the existence of a velocity-potential ϕ. Putting

$$\phi = f(r)\, e^{in\vartheta+iqt},$$

we get

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{n^2}{r^2} f = 0,$$

which gives

$$f(r) = \mathrm{A}r^n + \mathrm{B}r^{-n}.$$

As the velocity, as well inside as outside the cylinder-surface, must not be infinitely great, the potential inside the surface must be

$$\phi_1 = \mathrm{A}r^n e^{in\vartheta+iqt},$$

and outside

$$\phi_2 = \mathrm{B}r^{-n} e^{in\vartheta+iqt}.$$

Let the surface be expressed by

$$r - a = \zeta = \mathrm{C}e^{in\vartheta+iqt}.$$

For $r = a$ the following conditions must be satisfied :—

$$\frac{\partial \zeta}{\partial t} = -\frac{\partial \phi_1}{\partial r} = -\frac{\partial \phi_2}{\partial r} \quad . \quad . \quad . \quad . \quad (1) \qquad \text{and} \qquad p_1 - p_2 = \mathrm{T}\frac{1}{\mathrm{R}} \quad . \quad . \quad . \quad . \quad (2)$$

From (1) we get

$$\mathrm{B} = -\mathrm{A}a^{2n} \quad \text{and} \quad \mathrm{C} = i\mathrm{A}\frac{n}{q}a^{n-1},$$

from (2) we get

$$\rho_1 \frac{\partial \phi_1}{\partial t} - \rho_2 \frac{\partial \phi_2}{\partial t} + \mathrm{F}(t) = \mathrm{T}\left(\frac{1}{a} - \frac{\zeta}{a^2} - \frac{1}{a^2}\frac{\partial^2 \zeta}{\partial \vartheta^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Introducing in (3) the values found for ϕ_1, ϕ_2, and ζ, we get

$$q^2 = \frac{\mathrm{T}}{\rho_1 + \rho_2}\frac{n^3 - n}{a^3}.$$

In the above we have investigated the influence on the phenomenon in question of the viscosity of the liquid, the magnitude of the wave-amplitudes, and the inertia of the air.* Collecting the results found, we get the following formula for determination of the surface-tension, setting $n = 2$, as will be the case in the experiments :—

$$\mathrm{T} = \frac{(\rho_1 + \rho_2)\, k^2 a^3 c^2\, \mathrm{J}_2(iak)}{(3 + a^2k^2)\, iak\, \mathrm{J}'_2(iak)} \left[1 + 2\left(\frac{2\mu}{\rho c a^2 k}\right)^{3/2} + 3\left(\frac{2\mu}{\rho c a^2 k}\right)^2\right]\left(1 + \frac{37}{24}\frac{b^2}{a^2}\right).$$

* The corrections are to be considered as additive, since it can be shown that the wave-length also in the case of viscosity will be an even function of b/a.

Before proceeding to the experimental part of the investigation, we will yet consider a question which may be of interest for the discussion that follows.

In the jets produced in the experiments the velocity must be supposed to be greater in the middle of the jet than closer to the surface.

We can, however, in the following manner, get an idea of the rate at which the velocity-differences will be extinguished by the viscosity of the liquid. For this purpose we will consider a circular fluid-cylinder, in which each part moves with a velocity parallel to the axis of the cylinder, and in which the velocities of the different parts are only functions of the distance from the axis and the time.

Using the axis of the cylinder as Z-axis, we have with the same notation as above,

$$\alpha = 0, \quad \beta = 0, \quad \text{and} \quad w = f(r, t).$$

From the two first equations it follows that $\frac{\partial p}{\partial r} = 0$, and, as further, $p = \text{const.}$ for $r = a$, we get $p = \text{const.}$

Supposing $w = \phi(r)\, e^{-\epsilon t}$, the equation of motion

$$\mu \nabla w - \rho \frac{\mathrm{D}w}{\mathrm{D}t} = \frac{\partial p}{\partial z} \quad \text{gives} \quad \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\rho \epsilon}{\mu}\phi = 0,$$

of which the solution, subject to the condition to be imposed, when $r = 0$, is

$$\phi = c\mathrm{J}_0(kr), \quad \text{in which} \quad k^2 = \frac{\rho\epsilon}{\mu}.$$

The dynamical surface-condition gives $\left(\frac{\partial w}{\partial r}\right)_{r=a} = 0$; therefrom it follows that k must be a root of the equation

$$\mathrm{J}'_0(ka) = 0, \quad (k_0 = 0,\ k_1 a = \pi 1{\cdot}2197,\ k_2 a = \pi 2{\cdot}2330,\ k_3 a = \pi 3{\cdot}2383, \ldots).$$

The general expression for w is consequently

$$w = \Sigma c_n \mathrm{J}_0(k_n r)\, e^{-\frac{\mu}{\rho} k_n^2 t}.$$

We see that the term of the expression for w containing $\mathrm{J}_0(k_1 r)$ decreases much more slowly than the terms with higher index.

For the jets in question the term mentioned will furthermore be predominant already at the orifice, as $\partial w/\partial r$ must be supposed to have the same sign between 0 and a.

The velocity in a jet-piece must therefore be expressed with a high degree of approximation by

$$w = c_0 + c_1 \mathrm{J}_0(k_1 r)\, e^{-\epsilon t}, \quad \text{where} \quad \epsilon = \frac{\mu}{\rho}\left(\frac{\pi 1{\cdot}2197}{a}\right)^2,$$

t being the time which the jet-piece has taken to move from the orifice.

Production of the Jet.

The most important question in the experiments is to produce a jet which, while satisfying the suppositions made in the theoretical development, executes vibrations of a single type.

This demand, however, cannot be expected to be satisfied by a portion of a jet of liquid which lies at a short distance from the orifice. Apart from possible variations of the value of the surface-tension on account of the very rapid extension of the surface, it will be very difficult to obtain pure harmonic vibrations of the jet at this place, for this requires not only a definite form of the cross-section of the jet at the orifice, but also a definite velocity at every point of this section. While it might be possible to satisfy the first condition by suitable choice of the orifice (see p. 296, (38)), it would, no doubt, be very difficult to satisfy the last; among other reasons the velocity of the fluid will, for various causes, be greater in the middle of the jet than closer to the surface. It is, therefore, of great importance to produce a jet which is so stable that the vibrations can be examined at a considerable distance from the orifice where the viscosity of the liquid has had time to act.

A jet issuing from a hole in a thin plate is, however, not very stable, and therefore the jet rather rapidly falls into drops. If, however, drawn-out glass-tubes are used as orifice, very long and stable jets can be formed when the tube has a suitable shape.

In the experiments, jets were exclusively employed the qualities of which repeated themselves twice over the circumference.

The orifices of the glass-tubes employed were given an elliptic section by specially heating the tubes, before drawing them out, on two opposite sides. Twisting of the glass-tubes would produce a rotation of the jet about its axis, and the planes of vibration would not preserve the same direction at different distances from the orifice; to avoid such results it was necessary during the heating and drawing-out to have both ends of the tube fastened on slides which could be displaced along a metal-prism.

When the glass-tubes were drawn out and cut off, they were examined under a microscope, and only those whose orifice had a uniform elliptic section were used. After this the jets, which were formed by the tubes, were examined. The purpose of this examination, which will be mentioned later (p. 307), was to find out if the jet was symmetrical about two perpendicular planes passing through its axis.

It has been mentioned above that, because of the effect of viscosity, a portion of the jet will execute vibrations, which are in better conformity with those wanted the more removed it is from the orifice. It might here be of interest to illustrate this by an example.

As such can be employed an experiment carried out with tube I (see the table on p. 310).

The jet in question, which had a mean radius $a = 0\cdot0675$ cm. and a velocity $c = 425$ cm./sec., was so stable that the wave-length could be measured with great exactness up to a distance of about 35 cm. from the orifice. We will now examine such a jet at a distance of 30 cm. from the orifice.

The viscosity will firstly have the effect that an original difference in velocity at different points of the cross-section of the jet is rapidly extinguished.

The calculation on p. 298 shows that the differences mentioned must decrease approximately as $e^{-\epsilon t}$, where $\epsilon = \mu/\rho\,(\pi 1\cdot2197/a)^2$. Let, now, $a = 0\cdot0675$ and $\mu/\rho = 0\cdot0125$ (temperature 11·8° C.), we get $\epsilon = 40\cdot3$. Let, further, $t = \frac{30}{425}$, and we get $e^{-\epsilon t} = e^{-2\cdot844} = 0\cdot0582$. We see from this that the differences in velocity at the place in question must be about 17 times smaller than close by the orifice.

The viscosity will furthermore have the effect that also the waves on the surface of the jet tend to be of single types. We found above that the general form of the surface of the jet, considering the vibrations as infinitely small, can be expressed by

$$r = a + \Sigma b_n \cos(n\vartheta + \tau_n)\cos(k_n z + \gamma_n)\,e^{-\epsilon_n z},$$

where with approximation we have

$$\epsilon_n = \frac{\mu}{\rho}\frac{2n(n-1)}{ca^2}.$$

Let now $a = 0\cdot0675$, $\mu/\rho = 0\cdot0125$, $c = 425$ and $z = 30$, we get $e^{-\epsilon_2 z} = 0\cdot461$, $e^{-\epsilon_3 z} = 0\cdot098$, $e^{-\epsilon_4 z} = 0\cdot0096$, $e^{-\epsilon_5 z} = 0\cdot00043$, $e^{-\epsilon_6 z} = 0\cdot000009$, &c.

If now the form of the surface of the jet is close to the orifice,

$$r = a + b_2\cos 2\vartheta\cos k_2 z + b_3\cos 3\vartheta\cos k_3 z + b_4\cos 4\vartheta\cos k_4 z + \ldots,$$

the form of the surface will therefore be at a distance of 30 cm. from the orifice, approximately,

$$r = a + \tfrac{1}{2}\left(b_2\cos 2\vartheta\cos k_2 z + \tfrac{1}{5}b_3\cos 3\vartheta\cos k_3 z + \tfrac{1}{50}b_4\cos 4\vartheta\cos k_4 z + \tfrac{1}{1000}b_5\cos 5\vartheta\cos k_5 z + \ldots\right).$$

For the jet used, the term with $\cos 2\vartheta\cos k_2 z$ was already at the orifice quite predominant, and especially the quantities $b_3, b_5, \ldots$, were very small in proportion to b_2, as the jet at the examination mentioned was found to be very nearly symmetrical about two perpendicular planes through its axis.

We thus see that the jet in the experiment mentioned at a distance of 30 cm. from the orifice must have executed exceedingly pure vibrations.

In the experiments ordinary tap-water was used.

For the sake of the investigation it was important to get a jet which could run without variation (same velocity and temperature) as long as wanted. In order to give the water a suitable constant temperature, it was led from the tap through a long leaden spiral tube, placed in a water-bath, and a regulator connected with the gasburner heating the bath. In this way the temperature of the water could be kept constant at 0·01° C. as long as wanted.

The arrangement for keeping the pressure constant is shown in fig. 1. The water coming from the heating apparatus was led into a glass-bottle A, in which a constant water-level was maintained with help of an overflow B. From A the water was led down to the pressure-reservoir, consisting of two 5-litre glass-bottles C and D.

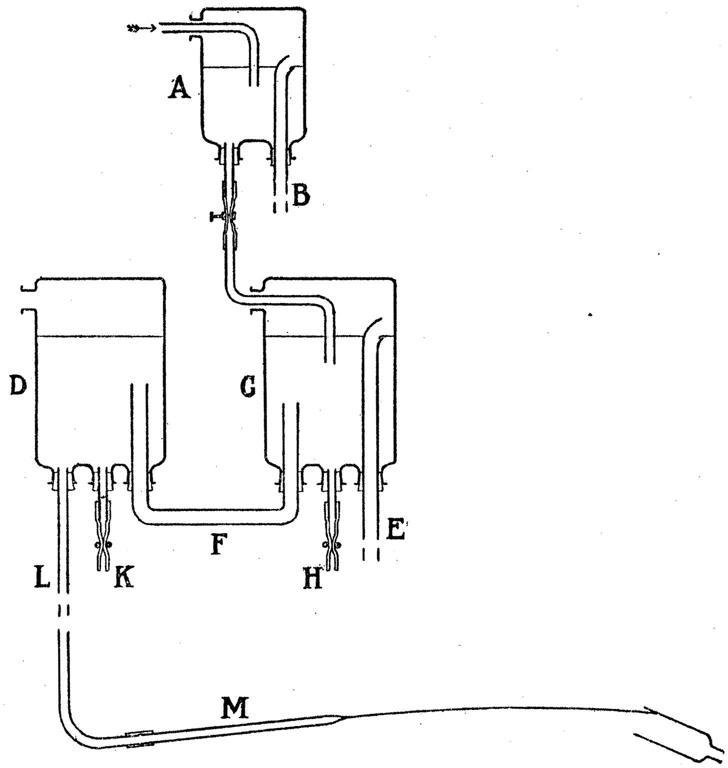


Fig. 1.

Inside C was placed an overflow E. C and D were connected by a wide bent-glass-tube F. H and K were two outlets through which the bottles could be emptied. From D the water was led down through a long glass-tube L to the jet-tube M. The whole arrangement was situated in a cellar, and the pressure-reservoirs as well as the jet-tube were supported by stone foundations. At the beginning of each experiment all the reservoirs and tubes were carefully cleaned and rinsed, whereupon the waterflow was adjusted so that a constant not particularly rapid flow ran through both the overflows.

With the arrangement mentioned, the water-surface in the bottle D was very steady and quite independent of the variations of the pressure in the supply pipe.

The temperature of the water was very near 12° C. in all experiments.

In order to calculate the surface-tension of a liquid, the following quantities had to be known :—(1) the density, ρ ; (2) the discharge per second, V ; (3) the velocity of the jet, c; (4) the mean radius of the jet, a (which four quantities are connected by the relation $V = \rho c \pi a^2$); (5) the wave-length, and, finally for the correction, (6) the amplitudes of the waves.

The density ρ of the tap-water used was at 12° found to be so near 1 (ρ = about 1·0001) that by putting $\rho = 1$ only errors far below the exactness of the experiment were made.

The measuring of the discharge presented no difficulty, and could be executed to 0·02 per cent. of its value.

*Determination of the Velocity of the Jet.**

When the jet is formed by a glass-tube, the velocity cannot be exactly calculated by the height of pressure on account of the friction in the tubes. In the present

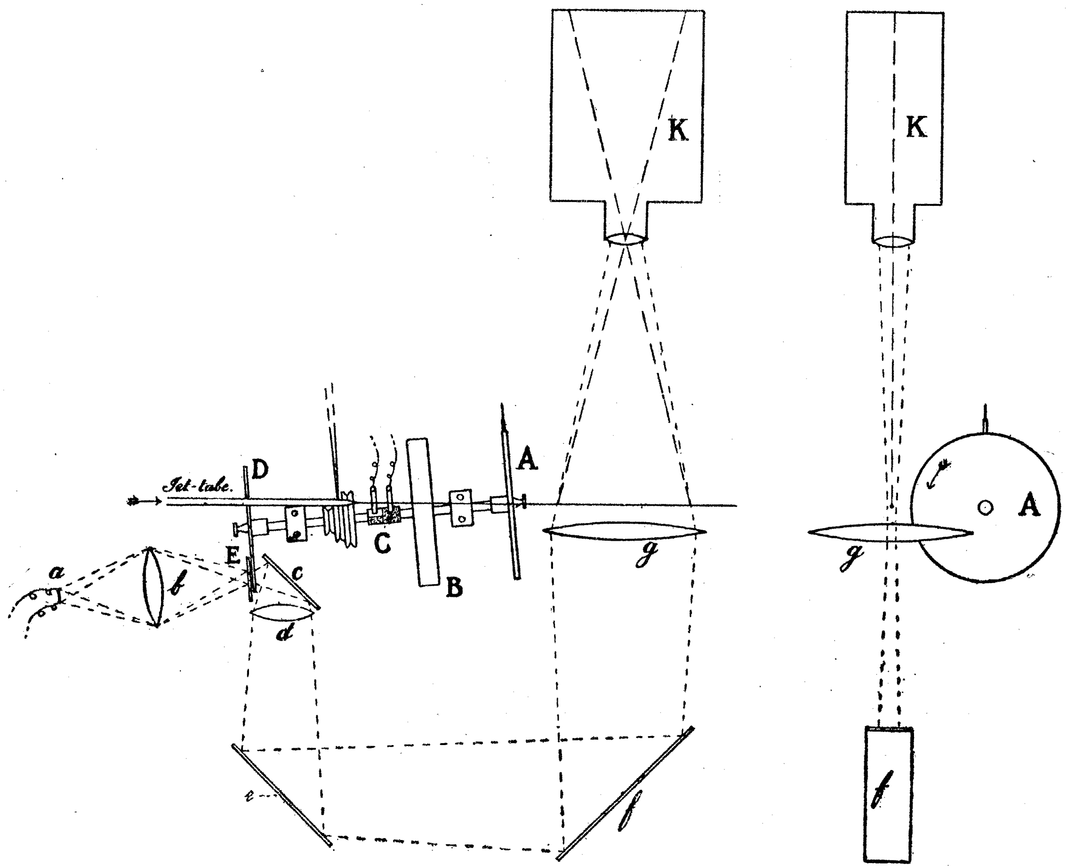


Fig. 2.

investigation a direct method was therefore used to measure the velocity of the jet, the main features of which were as follows: In a fixed point the jet was cut through, at constant time-intervals, by help of a sharp and thin knife, and at the

* A critical account of methods used in former investigations is to be found in the paper of P. O. PEDERSEN (*loc. cit.*, p. 352).

same time photographed instantaneously. Let the distance between two cuts, measured by help of the photographic plate, be a, and the time-interval be t, we have $c = a/t$, c being the velocity of the jet.

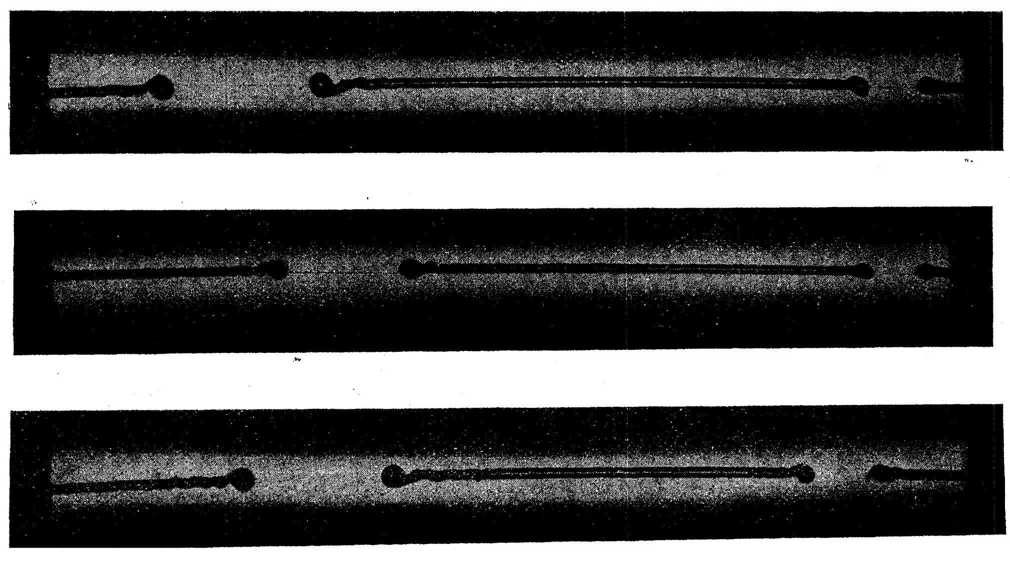
Fig. 2 shows the arrangement seen from above and from the side.

The rotation-apparatus ABCD executes the cutting of the jet and the opening and closing of the light. A is a metal disk, to the edge of which the knives were fastened in radial direction. The knives, made from ground needles, measured about 0·4 mm. in width and were about 0·03 mm. thick. The axis of the rotation-apparatus was not parallel to the jet, but formed a small angle with it, so that the knife cutting through the jet had the same velocity parallel to the axis of the jet as the water-particles.

D is a metal disk which has a radial slit close to the edge, which once at every revolution passes a corresponding slit in the screen E. The apparatus was driven by an electric motor, the speed of which could be regulated by means of an adjustable resistance, and, in order to make the velocity steady, the axis of the rotation-apparatus was provided with a small fly-wheel B. Further, to count the revolutions, the axis of the apparatus carried a contact C, which, completing the circuit of an electric current once at every revolution, marked a kymograph by help of an electro-magnet. The kymograph was also marked every second by another electromagnet.

abcdefg provided for the illumination of the jet. By help of a powerful lens-system b an image of the horizontal linear filament of a Nernst lamp a was formed on the slit of the screen E. The mirrors c, e, f, and the lenses d and g, thereupon formed a magnified image of the slit on the jet, and from the lens g all the light was finally directed into the camera K. In the figure the dotted lines show the limitation of the beam of light.

Every photograph was taken during about 12 seconds, which corresponded with about 600 revolutions of the apparatus with the following exposures of the plate. Some photographs are shown in the accompanying figure. (The direction of the jet is



from right to left.) We see how the ends of the jet set free by the knives very rapidly contract themselves into the corresponding jet-pieces, assuming a drop-like appearance.

The plates were measured by examining them pressed together with a glass-rule under a microscope, and reading on the scale the positions of the lines, perpendicular to the jet, touching the drop-like free ends of the jet-pieces. Thereupon the mean of the results found for the two ends of each cut was calculated, and the difference between the means from two succeeding cuts, divided by the magnification of the plate, was equated to the distance which the jet had moved while the rotation-apparatus had made a revolution. The conditions of the correctness of this were partly that the cuts moved independently of each other, partly that the ends of the jet contracted themselves equally into the respective jet-pieces during the time which a cut took to move from the one place where it was photographed to the other. That these conditions were satisfied appears, first, from the fact that the part of the jet-pieces placed midway between the cuts was completely undisturbed by the cutting of the jet (see the photographs), and, secondly, from the symmetrical forms of the ends of the jet facing each other.

The magnification of the plate was found by taking a photograph of a glass-rule placed directly under the jet.

The interval of time between the cuts was determined as a mean from the number of revolutions per second during the time of exposure; the photographic plate also giving a sort of mean of the single exposures, very great accuracy might be obtained in this way.

At each determination of the velocity of the jet photographs were taken for the sake of the control with different times of revolution of the rotation-apparatus.

The following table shows the result of an experiment by which four photographs were taken :—

Magnification of the photographs. f.	Distance between two cuts. a.	Rotations per second. n.	Velocity of the jet. $v = \frac{na}{f}$.
	cm.		cm./sec.
0·8624	8·37	40·19	390·0
0·8624	6·735	49·92	389·8
0·8624	6·845	49·15	390·1
0·8624	6·54	51·41	389·9

The values found show a very good agreement, the largest mutual deviation being less than 0·1 per cent.

Determination of the Wave-Length.

In the experiments, jets were used with so small wave-amplitudes that the wave-length could not be measured directly with sufficient accuracy either on the jet itself or on a photograph of the same.

The method used to determine the wave-length consisted in finding out the summits of the jet (the points where the tangent-planes were parallel to the axis of the jet), using the jet as an optical image-forming system.

Fig. 3 represents a horizontal jet-piece S (placed so that the two perpendicular planes of symmetry of the jet are respectively horizontal and vertical), a telescope T,

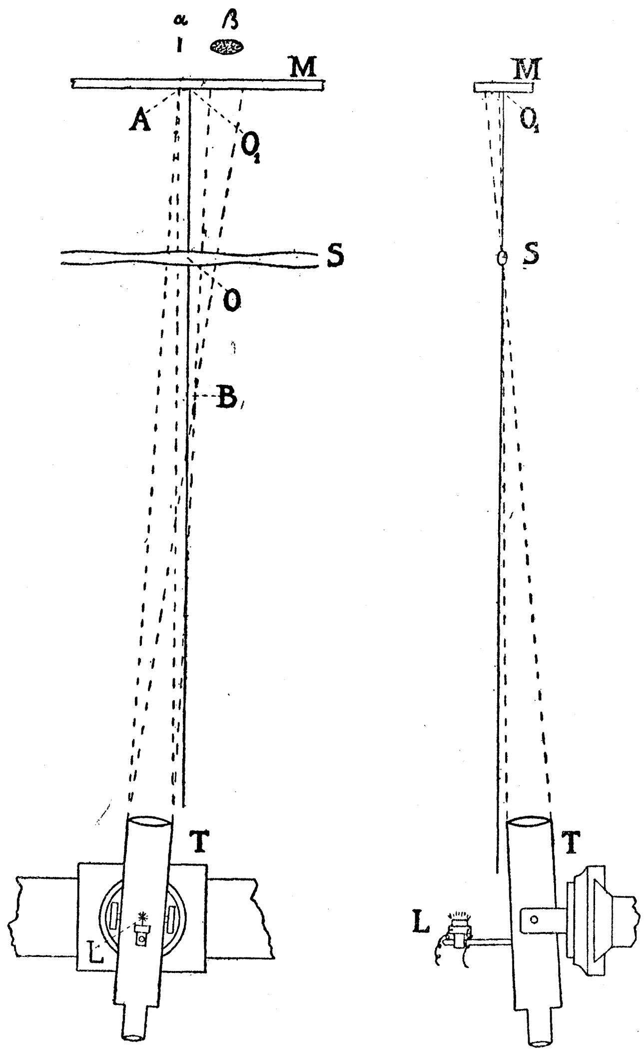


Fig. 3.

and a Nernst lamp L (the filament being vertical) fastened to the telescope in a position vertically over its axis, seen from above and from the side. OO_1 is a horizontal line, perpendicular to the jet, through a summit.

Seen from above, the jet acts as a lens with large focus-width, the front-surface of which will form a virtual image at A, and whose back-surface will form a real image, modified through the refraction during the double passage through the front-surface at B. Seen from the side, all the light reflected can, on account of the small diameter

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

of the jet, be considered as intersecting the jet-axis. If, now, the telescope is focussed for the distance TA, a small vertical bright line α, and a less bright, but sharply limited ellipse β, with horizontal great axis will therefore be seen in a dark field.

When the telescope is displaced parallel to the jet, the distance between the bright line and the ellipse will vary, and the telescope being brought into a position quite opposite a summit, the bright line will fall together with the minor axis of the ellipse.

Parallel to the jet was placed a fine glass-rule M, divided into $\frac{1}{10}$ mm., which divisions could be seen sharply in the field of the telescope, together with the bright lines mentioned.

The measurements were executed in such a manner that the telescope was partly displaced parallel to the jet and partly turned around a vertical axis, until the vertical spider-line fell together with the bright line at the same time as this halved the ellipse.

Fig. 4 shows the appearance of the telescope-field. Every time when such an adjustment was obtained, the position of the spider-line was read on the glass-scale. The adjustment and reading could be done with an accuracy of 0·01 mm.

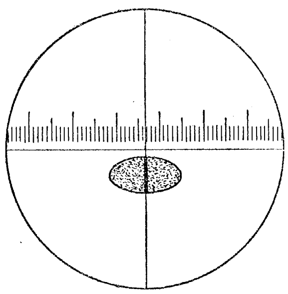


Fig. 4.

In the above we have supposed that the jet-axis was horizontal. If, on the contrary, the jet formed an angle with the horizontal plane—and this must be the case at certain places of the jet-piece examined on account of the curvature of the jet—the bright line and the minor axis of the ellipse will form the same angle with the vertical spider-line. If, however, care was taken in the arrangement that the centre of the vertical line from the middle of the filament to the telescope-axis was at the same horizontal height as the jet, the vertical plane through the telescope-axis will here, too, as a closer examination shows, be perpendicular to the vertical plane containing the jet and go through a summit when the vertical spider-line goes through the middle of the bright line at the same time as this falls together with the minor axis of the ellipse.

The circumstance that the wave-amplitudes on account of the viscosity of the liquid are decreasing in direction from the orifice has the effect that the distance OA between the focus-lines and the jet is not the same everywhere. While this fact is not of great importance when measuring the wave-lengths on a short jet-piece, it will, when measuring on very long jet-pieces (as in the table, p. 310) have the effect that the focussing of the telescope cannot be kept constant during the measuring, and the readings of the single summits could therefore, in this case, not be executed with quite as great accuracy as mentioned above.

The differences between the readings indicate the distances between the projections of the summits on a horizontal plane. Dividing the mentioned differences by cos α, α being the slope of the jet at the place in question, we therefore get the distances between the summits. These distances can be directly put equal to the wave-lengths sought, for, considering the jet-axis as a straight line, the wave-profile can, apart from possible irregularities, be expressed with great approximation by (see p. 296)

$$r = a + be^{-\epsilon z}\cos kz + \frac{1}{24}\frac{b^2}{a}e^{-2\epsilon z}\cos 2kz + \frac{1}{8}\frac{b^2}{a}e^{-2\epsilon z}.$$

Finding the summits z_n of this curve by putting $\partial r/\partial z = 0$, we get approximately

$$z_n - n\frac{\pi}{k} = -\frac{\epsilon}{k^2} - \frac{1}{6}\frac{b}{a}\frac{\epsilon}{k^2}e^{-\epsilon z_n}.$$

The first term on the right side is a constant, and with the values of ϵ, k, and b/a, which correspond to the experiments carried out, the second term is quite negligible as compared with the accuracy of the experiments.

P. O. PEDERSEN has also measured the wave-length on a jet with very small wave-amplitudes. The author mentions* that it has not been possible for him to produce jets with such regular vibrations that he could use a method for the determination of the wave-lengths which he describes and which in the main features is of the same nature as that described above. He therefore employed another method, which is, in the main, as follows: Illuminating the jet with a parallel beam of light, the rays twice refracted and once reflected form a wave-like image on a photographic plate, and as the amplitude of the image is much larger than that of the jet, the wave-length could be measured directly on the image. By this method the wave-length is determined as mean wave-length on a longer jet-piece. As however will appear from experiments, which will be described later, it is of the greatest importance to be able to determine the single wave-length with great enough accuracy for the variations of the wave-lengths to be examined.

The image-formation of the jet was also used in the examination of the jets mentioned on p. 299. If the tube was turned around its axis at the same time as the images were observed in the telescope, the appearances of these changed as the curvature of the profile of the jet seen from above gradually varied, the points A and B being displaced. The variations in the appearance of the images were most rapid in the moments in which the curvature of the profile was near to O. Every time the curvature became O, the points A and B fell together and a regular elliptic luminous spot without structure was seen in the telescope. To the tube was fastened a disk with a graduation, and every time the luminous spot mentioned appeared in the telescope during the revolution of the tube the graduation was read off. If the jet were symmetrical with respect to two perpendicular planes, the places read off must lie symmetrical on the circumference of the circle and also be the same at different distances from the orifice. This examination was very sensitive and it showed, too, that not all of the tubes examined satisfied the conditions to a sufficient

* PEDERSEN, *loc. cit.*, p. 368.

degree of accuracy. However, four satisfactory tubes were found. That the jets produced by these four tubes executed exceedingly pure vibrations appears also very distinctly from the measurement of the wave-length, which will be mentioned later.

In the experiments the jet had to be placed so that the two symmetrical planes were respectively horizontal and vertical. This was attained by turning the tube into a position midway between two of the above-mentioned readings.

The Photographing of the Jet.

To determine the magnitude of the amplitudes of the waves, magnified photographs of the jet were taken.

Using nearly monochromatic light, and a special limitation of the illuminating beams, the profile of the jet was brought to appear with very great sharpness on the photographic plates.

By help of an object-micrometer the diameter of the jet was measured at different places of the plate. The single diameters could in this way, on account of the sharpness of the outline, be measured with a relative exactness of about 0·03 per cent. of their value.

From the largest and smallest diameter of the jet ($2r_{\text{max.}}$ and $2r_{\text{min.}}$) the amplitude $\frac{b}{a} = \frac{r_{\text{max.}} - r_{\text{min.}}}{r_{\text{max.}} + r_{\text{min.}}}$ and the mean radius $a = \frac{1}{2}(r_{\text{max.}} + r_{\text{min.}})\left[1 - \frac{1}{6}\left(\frac{b}{a}\right)^2\right]$ [see p. 296 (40)] were determined. (On account of the decreasing of the amplitudes for $r_{\text{max.}}$ and $r_{\text{min.}}$ respectively the mean of two succeeding $r_{\text{max.}}$ and the $r_{\text{min.}}$ lying between were used.)

The value of the mean radius obtained in this way showed a very great conformity with the value which could be calculated by help of the discharge and the velocity of the jet, measured in the manner described above.

In order to show this, two experiments (executed with the tubes I and IV) will be mentioned, by which the mean radius of the jet was determined in both ways:—

TUBE I.

$r_{\text{max.}}$ 0·06929 cm., 0·06918 cm.	Discharge V = 6·274 cm.³/sec.
$r_{\text{min.}}$ 0·06549 cm.	Velocity c = 440·8 cm./sec.
$\frac{b}{a} = 0{\cdot}0278$, $a = 0{\cdot}06736$ cm.	$a = \sqrt{\frac{V}{\pi c}} = 0{\cdot}06731$ cm.

TUBE IV.

$r_{\text{max.}}$ 0·08263 cm., 0·08255 cm.	Discharge V = 7·862 cm.³/sec.
$r_{\text{min.}}$ 0·07777 cm.	Velocity c = 390·0 cm./sec.
$\frac{b}{a} = 0{\cdot}0301$, $a = 0{\cdot}08017$ cm.	$a = \sqrt{\frac{V}{\pi c}} = 0{\cdot}08011$ cm.

We see that the mean radii a, determined in the two ways, are very nearly the same (mutual deviation less than 0·1 per cent.). This conformity having been stated, the velocity-determination was omitted at the later experiments and a was determined by help of the photographs only, whereby the experiments became very much simplified.

The Results of the Experiments.

In the above we have described the methods used in the different measurements; it is further mentioned how it was possible, by the arrangement described on p. 301, to keep the pressure-height and temperature of the water constant during the comparatively long space of time taken to determine the discharges, the velocity, the mean radius, and the wave-length.

Before giving the results of the experiments we must, however, call attention to some special circumstances occurring in the determination of the wave-lengths sought, due to the fact that the wave-lengths found were not equal at different distances from the orifice. In order to show plainly what is meant by this, we shall commence with mentioning four experiments (one executed with each of the four tubes) carried out at a pressure-height of about 100 cm., in which the single wave-lengths were determined immediately outside the orifice and as far out on the jet as its stability permitted.

The results can be seen in the table overleaf.

As it will be seen, the differences between the readings are not constant, but increase until they reach a maximum, whereupon they slowly decrease again. The same can be seen from the table on p. 311, where the numbers in the column designated by "mean values" are calculated from the table overleaf by a simple adjustment.

The variation of the differences read off is, however, the result of many causes, among which are some the influence of which can be directly calculated. The first cause is the curvature of the jet, the effect of which is partly that the differences found are not equal to the real wave-lengths (see p. 306), partly that velocity and cross-section are not the same at different places of the jet-piece examined. The second cause is the decreasing of the wave-amplitudes, the influence of which appears from the equation (37) on p. 296. The column of the table on p. 311 designated by "corrected values" therefore contains wave-lengths, at different distances from the orifice, belonging to a horizontal jet which has the same velocity and cross-section as the jet examined on the horizontal place and which executes vibrations with infinitely small wave-amplitudes.

We see that the numbers in the last-mentioned column increase until they reach a maximum, whereupon they keep very nearly constant. This seems to show that all the causes of the variation of the wave-length, the influence of which is not corrected for, must originate in irregularities of the phenomenon which arise in the

Tube	I.		II.		III.		IV.	
Temperature	11·82° C.		11·73° C.		11·76° C.		11·80° C.	
Distance from the orifice of the horizontal part of the jet	26·3 cm.		29·4 cm.		28·9 cm.		34·6 cm.	
Discharge	6·100 cm.³/sec.		7·678 cm.³/sec.		7·720 cm.³/sec.		8·649 cm.³/sec.	
Mean radius of the jet on the horizontal place	0·06755 cm.		0·07554 cm.		0·07595 cm.		0·08010 cm.	
Readings on a horizontal rule carried out with an exactness of 0·005 cm.	cm.		cm.		cm.		cm.	
Orifice	0·0		0·0		0·0		0·0	
Summit I.	0·99		2·39		2·375		2·555	
		2·03		2·545		2·525		2·76
„ II.	3·02		4·935		4·90		5·315	
		2·125		2·56		2·56		2·765
„ III.	5·145		7·495		7·46		8·08	
		2·155		2·575		2·58		2·795
„ IV.	7·30		*10·07		*10·04		10·875	
		2·17		2·605		2·605		2·83
„ V.	*9·47		12·675		12·645		13·705	
		2·18		2·635		2·62		2·85
„ VI.	11·65		15·31		15·265		*16·555	
		2·195		2·65		2·63		2·87
„ VII.	13·845		17·96		17·895		19·425	
		2·215		2·65		2·64		2·885
„ VIII.	16·06		20·61		20·535		22·31	
		2·215		2·655		2·645		2·895
„ IX.	18·275		23·265		23·18		25·205	
		2·22		2·66		2·65		2·90
„ X.	20·495		25·925		25·83		28·105	
		2·22		2·66		2·655		2·90
„ XI.	22·715		†28·585		†28·485		31·005	
		2·225		2·655		2·655		2·90
„ XII.	24·94		31·24		31·14		†33·905	
		2·225		2·66		2·655		2·90
„ XIII.	†27·165		33·90		33·795		36·805	
		2·225		2·655		2·655		2·90
„ XIV.	29·39		36·555		36·45		39·705	
		2·22		2·65		2·65		2·895
„ XV.	31·61		39·205		39·10		42·60	
		2·225		2·65		2·65		2·895
„ XVI.	33·835		41·855		41·75		45·495	
The amplitude $\frac{b}{a}$ on the summits designated by * and † — *	0·0417		0·0699		0·0640		0·0382	
— †	0·0258		0·0472		0·0432		0·0276	

formation of the jet and which are rapidly extinguished. We see, however, that the influence of these irregularities on the wave-length is not insignificant, even at a considerable distance from the orifice. Thus, in the experiments mentioned, the wave-lengths are at a distance of 10 cm. from the orifice in the mean 2 per cent., and

Distance from the orifice in centimetres.	I.		II.		III.		IV.	
	Mean values.	Corrected values.	Mean values.	Corrected values.	Mean values.	Corrected values.	Mean values.	Corrected values.
5	2·14	2·148	2·55	2·553	2·54	2·545	2·76	2·782
10	2·185	2·189	2·59	2·591	2·595	2·597	2·815	2·829
15	2·210	2·211	2·640	2·638	2·625	2·624	2·850	9·859
20	2·221	2·221	2·654	2·650	2·642	2·639	2·880	2·884
25	2·225	2·224	2·658	2·653	2·652	2·648	2·896	2·897
30	2·224	2·224	2·658	2·654	2·656	2·652	2·900	2·899
35	—	—	2·656	2·653	2·654	2·652	2·901	2·899
40	—	—	2·651	2·652	2·650	2·652	2·898	2·898

at a distance of 20 cm., 0·3 per cent., smaller than is the wave-length at a distance of 30 cm. from the orifice; if, therefore, the wave-length at 10 cm. or at 20 cm. distance from the orifice had been used for the calculation of the surface-tension, a value respectively 4 per cent. and 0·6 per cent. too great would have been obtained.

The four experiments mentioned furthermore illustrate the influence of the viscosity on the phenomenon, the magnitude of the wave-amplitudes at two places of the jet with considerable mutual distance being measured (see the table on p. 310).

Putting $\frac{b}{a} = \mathrm{A}e^{-\epsilon z}$, we get for the four jets respectively

$$\epsilon = 0{\cdot}0271, \qquad \epsilon = 0{\cdot}0212, \qquad \epsilon = 0{\cdot}0213, \qquad \epsilon = 0{\cdot}0187.$$

In the above we have found (p. 289 (40))

$$\epsilon = \frac{\mu}{\rho}\frac{4}{ca^2}\left(1+\frac{11}{12}a^2k^2\right)\left[1-\frac{1}{2}\left(\frac{2\mu}{\rho ca^2k}\right)^{1/2}\right].$$

From this formula we get the following results for μ,

$$\mu = 0{\cdot}0131, \qquad \mu = 0{\cdot}0129, \qquad \mu = 0{\cdot}0130, \qquad \mu = 0{\cdot}0129.$$

We see that these values do not differ much from the most generally adopted value for μ, namely, $\mu = 0{\cdot}0125$ (temperature, 11·8° C.). That they, however, are all greater suggests, perhaps, a very small superficial viscosity.

The correction of the formula to calculate the surface-tension, due to the effect of the viscosity on the wave-length, is, according to the equation (41), on p. 289, determined by the coefficient

$$1+2\left(\frac{2\mu}{\rho ca^2k}\right)^{3/2}+3\left(\frac{2\mu}{\rho ca^2k}\right)^2.$$

By introducing the values for μ, ρ, c, a, and k, which correspond to the experiments carried out, this correction becomes very small, about 0·1 per cent.*

As to the calculated correction for the influence of the finite wave-amplitudes, it may be mentioned that the values of the surface-tension in the table of the experiments on p. 313, which has been calculated according to the formula (37) on p. 296, does not show any systematic deviation due to the wave-amplitude.

As, however, the correction for the value of the wave-amplitude in the experiments carried out is rather small (from 0·10 per cent. to 0·33 per cent.), the agreement mentioned is not adapted to give an experimental verification of the formula theoretically developed. It may here be remarked that P. O. PEDERSEN (*loc. cit.*, p. 371) has experimentally investigated the influence of the value of the wave-amplitudes upon the calculated values of the surface-tension and has found results, by using greater wave-amplitudes, which can be shown to be in very good agreement with the formula in question.

In the other experiments the wave-length was measured only on a shorter jet-piece, which, however, was so far from the orifice that the value of the wave-lengths had become constant.

As an example of such a measurement, an experiment with tube I may be mentioned, which was carried out with a pressure-height of about 70 cm.

In the table below are quoted two sets of readings, obtained in succession, with their differences.

Readings.	Difference.	Readings.	Difference.
cm.	cm.	cm.	cm.
1·819		1·818	
	1·796		1·797
3·615		3·615	
	1·798		1·800
5·413		5·415	
	1·799		1·800
7·212		7·215	
	1·801		1·799
9·013		9·014	
	1·799		1·799
10·812		10·813	
	1·796		1·797
12·608		12·610	

The horizontal place of the jet was at a distance of 21·5 cm. from the orifice, which corresponded to a reading on the glass-rule of 7·5 cm. Corrections of the readings

* The smallness of the correction is due to the small coefficient of viscosity ($\mu = 0{\cdot}0125$) and the great surface-tension (T = 74) of water. The correction mentioned can, however, become quite considerable for liquids in which these quantities have other values; if, for example, aniline ($\mu = 0{\cdot}062$, T = 44) was used, the correction would have been more than 1 per cent. by corresponding experiments.

have not been introduced, as they were in this case very small; thus the correction for the decreasing of the wave-amplitudes would become completely imperceptible on account of the small value of the amplitudes. Besides, the very small decrease of the utmost differences corresponds to the one to be expected on account of the curvature of the jet.

We see that the wave-lengths are very constant, and that the jet at the place in question must have executed exceedingly pure vibrations, because only a small deviation from pure harmonic vibrations must involve considerable irregularities of the differences between the readings.

The other experiments carried out show results very similar to those described here. It may be remarked that, *à priori*, we could expect exceedingly pure vibrations of the jets from all the four tubes. The circumstance that the regular variation of the differences between the readings commenced directly past the orifice (see the table on p. 310) shows that already at this point we had to do with what was very nearly a single vibration only, and, as mentioned on p. 300, the vibrations of the jet must be much purer at a considerable distance from the orifice than close to it.

The table below contains the result of all the experiments carried out. The surface-tension is calculated according to the following equation (see p. 297):—

$$T_{12} = \frac{(\rho_1+\rho)\, k^2c^2a^3 J_2(iak)}{(3+a^2k^2)\, iak J'_2(iak)} \left[1+2\left(\frac{2\mu}{\rho c a^2 k}\right)^{3/2}\right]\left(1+\frac{37}{24}\frac{b^2}{a^2}\right)[1 \div 0{\cdot}002\,(12-t)].$$

Tube.	Temperature.	Discharge.	Mean radius.	Wave-length.	Amplitude.	T_{12}.
	° C.	cm.³/sec.	cm.	cm.		dyne/cm.
I.	11·8	6·100	0·06755	2·225	0·026	73·24
I.	11·4	5·608	0·06758	2·039	—	73·41
I.	11·3	4·965	0·06767	1·800	—	73·34
II.	11·7	7·678	0·07554	2·658	0·046	73·01
II.	11·2	7·076	0·07567	2·443	—	72·98
II.	11·4	6·272	0·07587	2·154	—	73·26
III.	11·8	7·720	0·07595	2·656	0·042	73·45
III.	11·4	6·290	0·07604	2·157	—	73·28
IV.	11·8	8·649	0·08010	2·901	0·027	73·21
IV.	11·9	7·984	0·08014	2·677	—	73·09
Mean value of the experiments with Tube I . .						73·33
,, ,, ,, ,, II . .						73·08
,, ,, ,, ,, III . .						73·37
,, ,, ,, ,, IV . .						73·15
Mean value of all experiments . .						73·23

We see that the mutual agreement between the single determinations is very good (the greatest deviation from the mean value being about 0·35 per cent.).

It may be remarked that in the values found for T_{12} no indication of a distinct influence originating either from the variation of the diameter of the jet, or of the discharge, or of the amplitudes of the waves can be found.

In all the experiments mentioned, tap-water was used. An investigation was, however, carried out to see if a different result would be obtained by using distilled water instead of tap-water. For this purpose two large reservoirs were filled respectively with distilled and tap-water. After the contents of the reservoirs had assumed the same temperature, measurements of wave-lengths in exactly the same conditions were undertaken on a jet of each of the two sorts of water by connecting first the one reservoir and then the other with the glass-bottle A, fig. 1, by a siphon. The experiment, which was repeated several times, showed that no sensible difference was to be found between the two jets.

This result was also to be expected from previous investigations on the surface-tension of water.

Now proceeding to compare the value found here with values found by previous determinations, we shall not try to give a complete account of the very extensive literature on this subject. The table opposite contains only the results of a few of the investigations of later years, which are generally considered the most important for the estimate of the value of the surface-tension.

The table shows rather considerable deviations between the values found by the different investigators. As an explanation of these deviations, the question of the purity of the surface has been among the most prominent, relying on the fact that the tension of a water-surface may decrease very considerably when the surface becomes contaminated with even an extremely small amount of foreign substances. This circumstance, however, does not seem sufficient to explain the deviations among the values found by authors who have used the same method for purifying the surface (*e.g.*, GRUNMACH and KALÄHNE; FORCK and ZLOBICKI).

The fact that a number of authors (*e.g.*, VOLKMANN, DORSEY, FORCK) who have worked with different methods have found such exceedingly good conformity among the results of their single experiments after all seems to show that the surface-tension of a carefully purified surface is a very constant quantity. This assumption is further confirmed by the circumstance that several authors (KALÄHNE, DORSEY, &c.) have not found any sensible diminishing of the surface-tension during the time of the experiment.

The results of the investigations by Miss A. POCKELS,* Lord RAYLEIGH,† and F. NANSEN‡ on the influence of contaminations upon the tension of a water-surface seem also highly to point in this direction.

In consequence of the above-mentioned, it therefore seems that a great deal of the

* A. POCKELS, 'Nature,' XLIII., p. 437; XLVI., p. 418; XLVIII., p. 152. 'Ann. d. Phys.,' VIII., p. 854.

† RAYLEIGH, 'Phil. Mag.' XLVIII., p. 321, 1899.

‡ F. NANSEN, 'Norweg. North Polar Exped. Scient. Results,' 10, 1900.

Authority.	Publication.	Method.	T_{12}.*
WEINSTEIN	'Metr. Beitr. d. K. Norm. Aich-Komm.,' VI., 1889	Capillary tubes	73·53
GOLDSTEIN	'Ztschr. Phys. Chem.,' V., p. 233, 1890	,,	73·82
RAMSAY and SHIELDS	'Ztschr. Phys. Chem.,' XII., p. 433, 1893	,,	71·67
QUINCKE	'WIED. Ann.,' LII., p. 1, 1894	,,	73·3–77·8
VOLKMANN	'WIED. Ann.,' LVI., p. 457, 1895	,,	73·72
DOMKE	'Wiss. Abh. d. K. Norm. Aich-Komm.,' III., 1902	,,	73·92
GRABOWSKY	'Diss.,' Königsberg, 1904	,,	73·71
SENTIS	'Thèse,' Grenoble, 1897	Capillary tubes (virtual)	74·24
HALL	'Phil. Mag.' (5), XXXVI., p. 385, 1893	Weighing of the tension	73·90
FORCK	'Ann. d. Phys.,' XVII., p. 744, 1905	Pressure in air-bubbles	77·25
ZLOBICKI	'Rozpr. Akad. Kraków,' S. 3, T. VI., A, p. 181, 1906	,, ,,	73·70
RAYLEIGH	'Phil. Mag.,' XXX., p. 386, 1890	Capillary ripples (advancing waves)	74·88
DORSEY	'Phil. Mag.,' XLIV., p. 369, 1897	,, ,,	74·08
WATSON	'Phys. Rev.,' XII., p. 257, 1901	,, ,,	75·15
KOLOWRAT-TSCHERWINSKI	'I. d. Russ. Phys.,' XXXVI., p. 265, 1904	,, ,,	73·22
KALÄHNE	'Ann. d. Phys.,' VII., p. 440, 1902	,, (standing waves)	74·67
GRUNMACH	'Wiss. Abh. d. K. Norm. Aich-Komm.,' III., 1902	,, ,,	76·35
BRÜMMER	'Diss.,' Rostock, 1903	,, ,,	75·39
LOEWENFELD	'Diss.,' Rostock, 1904	,, ,,	75·78
PEDERSEN	'Trans. Roy. Soc.,' A 207, p. 341, 1907	Jet-vibration	74·76

* From the papers in which the surface-tension is not given at 12° C., T_{12} is calculated by means of the formula $T_t = T_0 (1 - 0{\cdot}0020t)$ (the temperature-coefficient being known with sufficient accuracy for this purpose).

deviations in question must be explained not by real differences of the surface-tension, but by the methods used in measuring this tension.

We now proceed to consider more closely some of the investigations mentioned, and compare the results with that found in the present paper.

We will commence with P. O. PEDERSEN'S investigations, as his determination of the surface-tension of water is executed by the same method (jet-vibration) as that used by the author. PEDERSEN finds, as the table shows, a value which is considerably greater (about 2 per cent.) than the value here found. As, however, PEDERSEN has not examined the variations of the wave-length, but only determined the wave-length as mean wave-length on a jet-piece at a comparatively short distance from the orifice, the cause of the difference between the value found by PEDERSEN and the author may be that PEDERSEN probably has used too small a value for the wave-length (see p. 311).

Among the other methods to determine the surface-tension, the capillary-tube method and the method of capillary ripples are those mostly used and generally considered the most important.

Among the investigations carried out by the former methods, VOLKMANN'S must be especially mentioned on account of the excellent agreement between the single experiments which he has obtained, taking great care in the measurement of the dimensions of the tubes and in their purification. This agreement, being independent of the dimensions of the tubes and of the nature of the glass, seems to have taken away the foundations of the criticism of the results which the capillary-tube method can give. VOLKMANN finds, as is seen, a value which lies rather near the author's, the difference being, however, about 0·7 per cent.

As is to be seen from the table, a great number of investigations have recently been executed by the method of capillary ripples. We see that the values found by this method are generally higher than the value found in this paper. The mutual conformity between the results of the different investigations is however not very great. In the author's opinion this disagreement depends on the fact that in many cases the conditions of the experimental investigations do not sufficiently correspond to the assumptions on which the theoretical development rests; in what follows, an attempt has been made to show what is meant by this.

The experiments executed by the method mentioned can be divided into two groups, according to advancing rectilinear waves, produced by help of the vibrations of a glass plate fastened to one prong of a tuning-fork, or standing waves formed by interference between two systems of advancing circular waves, generated by two pins fastened to both prongs of a tuning-fork, being used.

Among the authors who have used the former method, only DORSEY and KOLOWRAT-TSCHERWINSKI seem to have examined the magnitude of the wave-length at different distances from the generator. Both of the investigators found considerable irregularities near the generator, the wave-length here being dependent on the distance from the plate and first becoming constant at a greater distance from this. The authors mentioned, being aware of this fact, used for calculating the surface-tension only the length of waves which were at a certain distance from the glass-plate (Dorsey 4 cm., and Kolowrat-Tscherwinski 8 cm.). As the wave-length near to the glass-plate was larger than further out, this may explain the fact that Dorsey and especially Kolowrat-Tscherwinski have found lower values than other investigators who have used the same method, but as it seems have not taken precautions in this direction.

The other method, using the standing waves, suffers, as also Kolowrat-Tscherwinski remarks, from certain defects, because the measuring of the wave-length taking place on the straight line which connects the above-mentioned pins, those waves only can be examined which are at so short a distance from the pins that there is no security of the phenomenon being sufficiently regular. On account of this the results found by this method do not seem to be very reliable, especially the very high values of the surface-tension, and the great deviations between the result of the single experiments found by Grunmach, Brümmer, and Loewenfeld may probably be explained by the very small distance (1·8 cm.) between the pins used by these investigators. Kalähne, who employs the same method, but has a distance of 7 cm. between the pins, also finds a value considerably lower and with a much better mutual conformity than the above-named investigators.

In consequence of these considerations it does not seem necessary for the author to conclude that the method of capillary ripples in reality gives a value essentially higher than the one found by the method described in this paper.

Conclusions.

In the present determination of the surface-tension of water the method of jet-vibration proposed by Lord Rayleigh is used; this method has the fundamental advantage that a perfectly fresh new-formed surface can be examined.

In the first part of this investigation it is shown how Lord Rayleigh's theory of infinitely small vibrations of a jet of non-viscid liquid can be supplemented by corrections for the influence of the finite amplitudes as well as for the viscosity.

In the experimental part of this investigation an attempt has been made to show how in a simple manner it seems to be possible to secure that the jet-piece used for the measurement satisfies the assumptions on which the theoretical development rests.

As the final result of his experiments the author finds the surface-tension of water at 12° C. to be 73·23 dyne/cm.